A Bayesian problem of testing two simple hypotheses for a Brownian bridge

Dmitrii Lisovskii

Lomonosov Moscow State University

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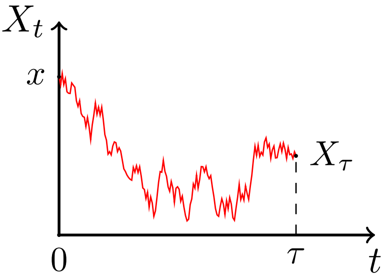
1. Introduction

Sequential hypotheses testing

Given two hypotheses about some process X to distinguish:

$$Hypothesis\ H_0 \qquad Hypothesis\ H_1$$



$\Leftarrow$ Some decision rule (τ, d)

- τ is an $(\mathscr{F}_t^X)_{t\geqslant 0}$-adapted st. time, where $\mathscr{F}_t^X = \sigma\{X_s, s \leqslant t\}$
- d is an $\mathscr{F}_\tau$-measurable random variable taking two values corresponding to the hypothesis to accept

Each procedure of sequential testing consists of the decision rule (τ, d).

- How to choose (τ, d)?

$$R(\tau, d) = \mathsf{E}\left[c\tau + W(d, \dots)\right] \to \inf_{(\tau,d)},$$

where $c > 0$ is some constant interpreted as a payment for the observations and $W(\dots)$ is responsible for the penalties because of a wrong terminal decision.

- How to solve?

$$\textit{Sequential testing problem}$$
$$\Updownarrow$$
$$\textit{Optimal stopping problem}$$
$$\Updownarrow$$
$$\textit{Free-boundary problem}$$

2. Problem formulation for a Brownian bridge

Model

Given an observable process

$$X_t = \mu\theta t + B_t^0, \quad 0 \leqslant t \leqslant 1,$$

B_t^0 being the unique strong solution to the following SDE

$$dB_t^0 = \frac{-B_t^0}{1-t}dt + dB_t, \quad B_0^0 = 0, \quad 0 \leqslant t < 1$$

- All processes and random variables are considered on some probability-statistical space $(\Omega; \mathscr{F}; \mathsf{P}_\pi, \pi \in [0,1])$
- B_t is a standard Wiener process, B_t^0 is a standard Brownian bridge process, $\mu \neq 0$ is some known constant, θ is a random variable s.t. $\mathsf{P}_\pi(\theta = 1) = \pi$ and $\mathsf{P}_\pi(\theta = 0) = 1 - \pi$
- B_t (or B_t^0) and θ are independent and θ cannot be observed directly but through the process X_t

Aim

We would like to test sequentially two simple hypotheses about the presence of a drift coefficient:

$$H_0\colon \theta = 0 \quad \text{and} \quad H_1\colon \theta = 1$$

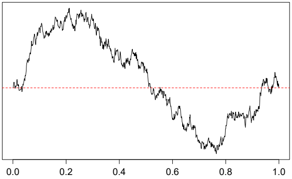


Figure 1: $X_t = B_t^0$

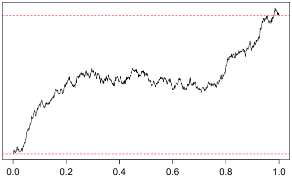


Figure 2: $X_t = \mu t + B_t^0$

Risk criterion

We will say that the decision rule is optimal if it minimizes the risk

$$V(\pi) = \inf_{(\tau,d)} \mathsf{E}_\pi \left(c\tau + a\mathbb{1}(d=0,\theta=1) + b\mathbb{1}(d=1,\theta=0)\right),$$

where E_π denotes the expectation w.r.t. measure P_π, $c, a, b > 0$.

Using standard technique (A. N. Shiryaev (1963)) one can show that the initial problem can be reduced to the optimal stopping problem

$$V(\pi) = \inf_\tau \mathsf{E}_\pi[c\tau + a\pi_\tau \wedge b(1-\pi_\tau)] \left(= \inf_\tau \mathsf{E}_\pi\left[c\tau + G(\pi_\tau)\right]\right)$$

$$d^* = \begin{cases} 1, & \text{if} \quad \pi_{\tau^*} \geqslant r \\ 0, & \text{if} \quad \pi_{\tau^*} < r \end{cases}$$

for the aposteriori probability process $\pi_t = \mathsf{P}_\pi\left(\theta = 1 | \mathscr{F}_t^X\right)$, $0 \leqslant t \leqslant 1$, with $\mathsf{P}_\pi(\pi_0 = \pi) = 1$ and $r = b/(a+b)$.

Remark: $G(r) = \max_{\pi \in [0,1]} (a\pi \wedge b(1-\pi))$

3. Results

Theorem

1. The optimal decision rule is given by the pair (τ^*, d^*) with

$$\tau^* = \inf\{0 \leqslant t \leqslant 1 \colon \pi_t \notin (g_0(t), g_1(t))\},$$

$$d^* = \begin{cases} 1 \text{ (accept } H_1), & \text{if } \pi_{\tau^*} = g_1(\tau^*) \\ 0 \text{ (accept } H_0), & \text{if } \pi_{\tau^*} = g_0(\tau^*) \end{cases}$$

where the boundaries (g_0, g_1) can be characterized as a unique solution to the system of non-linear integral equations ($i = 0, 1$)

$$c \sum_{j=0}^{1} (-1)^{j+1} \int_0^{1-t} \mathsf{P}_{t, g_i(t)} \left(\pi_{t+u}^{g_i(t)} \leqslant g_j(t+u) \right) du = a g_i(t) \wedge b(1 - g_i(t))$$

2. The explicit expression for these probabilities is given by

$$
\begin{aligned}
&\mathsf{P}_{t,g_i(t)}\left(\pi_{t+u}^{g_i(t)} \leqslant g_j(t+u)\right) = \\
&\qquad g_i(t)\Phi\left(\frac{\sqrt{1-t}\sqrt{1-t-u}}{\mu\sqrt{u}}\right)\times \\
&\qquad \ln\left(\frac{1-g_i(t)}{g_i(t)}\frac{g_j(t+u)}{1-g_j(t+u)}-\frac{\mu\sqrt{u}}{2\sqrt{1-t}\sqrt{1-t-u}}\right)+ \\
&\qquad (1-g_i(t))\Phi\left(\frac{\sqrt{1-t}\sqrt{1-t-u}}{\mu\sqrt{u}}\right)\times \\
&\qquad \ln\left(\frac{1-g_i(t)}{g_i(t)}\frac{g_j(t+u)}{1-g_j(t+u)}+\frac{\mu\sqrt{u}}{2\sqrt{1-t}\sqrt{1-t-u}}\right)
\end{aligned}
$$

3. The optimal pair of boundaries (g_0, g_1) has the following properties:

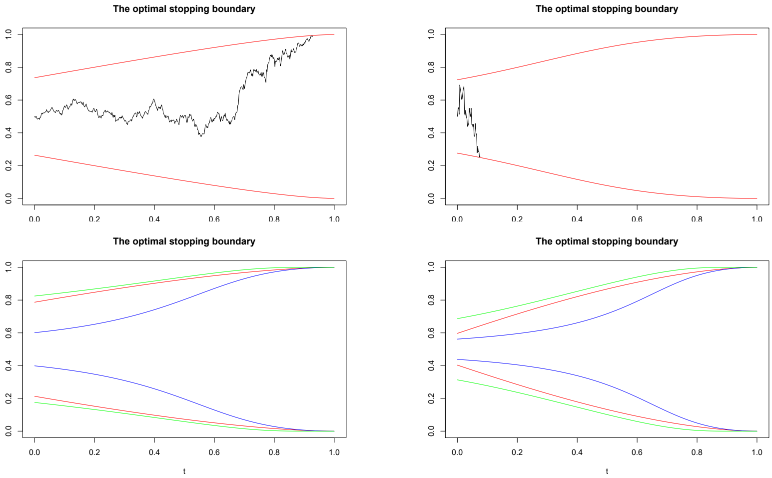
$$g_0\colon [0,1] \to [0,1] \quad \text{is decreasing and} \quad g_0(1) = 0$$
$$g_1\colon [0,1] \to [0,1] \quad \text{is increasing and} \quad g_1(1) = 1$$

Besides, the following inequalities are true

$$m_0\left(\frac{t}{1-t}\right) \leqslant g_0(t) \leqslant M_0\left(\frac{t}{1-t}\right) < r$$
$$r < M_1\left(\frac{t}{1-t}\right) \leqslant g_1(t) \leqslant m_1\left(\frac{t}{1-t}\right),$$

Provided $a = b$, we have $\widetilde{M}_1(t) = 1 - \widetilde{M}_0(t)$ and $\widetilde{m}_1(t) = 1 - \widetilde{m}_0(t)$ and the asymptotic behaviour as $t \uparrow 1$ is

$$\widetilde{m}_0(t) = \frac{c}{a}\frac{\sqrt{1+8/\mu^2}-1}{\sqrt{1+8/\mu^2}+1}e^{-1/(1-t)} + o(e^{-1/(1-t)}),$$
$$\widetilde{M}_0(t) = \frac{2c}{\mu^2 a}(1-t)^2 + o((1-t)^2).$$


The optimal stopping boundary
The optimal stopping boundary
The optimal stopping boundary
The optimal stopping boundary
0.0
0.2
0.4
0.6
0.8
1.0
t
t

4. Some known results

1. The Wiener sequential testing problem with finite horizon (Gapeev and Peskir, 2003)

- Model:

$$X_t = \mu\theta t + B_t, \quad 0 \leqslant t \leqslant 1$$

- All processes and random variables are considered on some probability-statistical space $(\Omega; \mathscr{F}; \mathsf{P}_\pi, \pi \in [0,1])$
- B_t is a standard Wiener process, θ is a random variable such that $\mathsf{P}_\pi(\theta = 1) = \pi$ and $\mathsf{P}_\pi(\theta = 0) = 1 - \pi$, $\mu \neq 0$ is some known constant
- B_t and θ are independent and cannot be observed directly, but through the process X_t

- Goal:

$$H_0 \colon \theta = 0 \qquad H_1 \colon \theta = 1$$

- Risk criterion:

$$V(\pi) = \inf_{(\tau,d)} \mathsf{E}_\pi \left[c\tau + a\mathbb{1}(d=0,\theta=1) + b\mathbb{1}(d=1,\theta=0) \right]$$

$$\Downarrow$$

$$V(\pi) = \inf_{\tau} \mathsf{E}_\pi \left[c\tau + a\pi_\tau \wedge b(1-\pi_\tau) \right]$$

$$d^* = \begin{cases} 1, & \text{if} \quad \pi_{\tau^*} \geqslant r \\ 0, & \text{if} \quad \pi_{\tau^*} < r \end{cases}$$

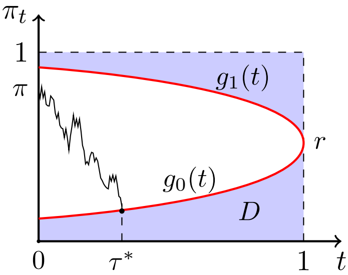
for the aposteriori probability process $\pi_t = \mathsf{P}_\pi \left(\theta = 1 | \mathscr{F}_t^X \right)$, $0 \leqslant t \leqslant 1$, with $\mathsf{P}_\pi \left(\pi_0 = \pi \right) = 1$.

- Solution:

The optimal decision rule is given by the pair (τ^*, d^*), where

$$\tau^* = \inf\{0 \leqslant t \leqslant 1 \colon \pi_t \notin (g_0(t), g_1(t))\},$$

$$d^* = \begin{cases} 1 \text{ (accept } H_1\text{)}, & \text{if } \pi_{\tau^*} = g_1(\tau^*) \\ 0 \text{ (accept } H_0\text{)}, & \text{if } \pi_{\tau^*} = g_0(\tau^*) \end{cases}$$



2. Chernoff's problem (Zhitlukhin and Muravlev, 2012)

- Model:

$$X_t = \mu t + B_t, \quad t \geqslant 0,$$

with $\mu \sim \mathcal{N}(\mu_0, \sigma_0^2)$ independent of $B = (B_t)_{t \geqslant 0}$ being the standard Wiener process.

- Goal:

$$H_0 \colon \mu > 0 \qquad H_1 \colon \mu \leqslant 0$$

- Risk criterion:

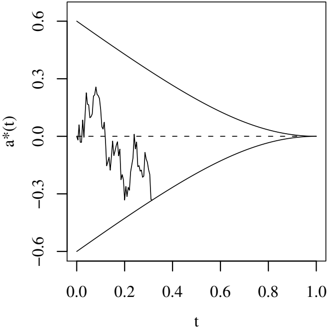
$$R(\tau, d) = \mathsf{E}\left[c\tau + k|\mu|\mathbb{1}(d \neq \operatorname{sgn}(\mu))\right] \to \inf_{(\tau,d)},$$

where $c, k > 0$ are some fixed constants, d is a r.v. taking values $\{-1, 1\}$ only, and $\operatorname{sgn}(0) = -1$.

- Solution:

$$\tau^* = \frac{\tau_W^*}{\sigma_0^2(1-\tau_W^*)}, \qquad d^* = \operatorname{sgn}\left(X_{\tau^*} + \frac{\mu_0}{\sigma_0^2}\right)$$

$$\tau_W^*(\mu_0, \sigma_0) = \inf\left\{0 \leqslant t \leqslant 1\colon \left|W_t + \frac{\mu_0}{\sigma_0}\right| \geqslant a_{\sigma_0}(t)\right\}$$



5. Scheme of the Proof

Step 1: dynamic of the aposteriori probability process

The general Bayes formula says that the aposteriori probability process can be expressed through the density process $\varphi_t = \frac{dP_1}{dP_0}\left(\mathscr{F}_t^X\right)$ as follows:

$$\pi_t^\pi = \frac{\frac{\pi}{1-\pi}\varphi_t}{1 + \frac{\pi}{1-\pi}\varphi_t}.$$

Here $\mathsf{P}_0(\cdot) = \mathsf{P}_\pi(\,\cdot\,|\,\theta = 0)$ and $\mathsf{P}_1(\cdot) = \mathsf{P}_\pi(\,\cdot\,|\,\theta = 1)$.

Due to the local absolute continuity (Liptser&Shiryaev) of measures P_0 and P_1, we see that φ_t admits the representation

$$\varphi_t = \exp\left(\int_0^t \frac{\mu}{1-s}dX_s + \frac{\mu}{2}\int_0^t \frac{2X_s - \mu}{(1-s)^2}ds\right)$$

$$\Longrightarrow d\varphi_t = \varphi_t\left(\frac{\mu}{1-t}dX_t + \frac{\mu X_t}{(1-t)^2}dt\right)$$

Hence, it is easy to check that

$$d\pi_t = \pi_t(1-\pi_t)\frac{\mu}{1-t}dX_t + \left(\pi_t(1-\pi_t)\frac{\mu X_t}{(1-t)^2} - \pi_t^2(1-\pi_t)\frac{\mu^2}{(1-t)^2}\right)dt$$

- Difficulty: X_t is not a diffusion-type process but an Itô one

$$dX_t = \frac{\mu\theta - X_t}{1-t}dt + dB_t$$

- Solution: the usage of an innovation process $\bar{B}_t$, i.e.

$$\bar{B}_t = X_t - \int_0^t \frac{\mu\pi_s - X_s}{1-s}ds$$

where $(\bar{B}_t, \mathscr{F}_t^X)$ appears to be a Bm and $\mathscr{F}_t^X = \mathscr{F}_t^{\bar{B}}$ for all $t \in [0,1]$.

$$\Longrightarrow dX_t = \frac{\mu\pi_t - X_t}{1-t}dt + d\bar{B}_t, \quad d\pi_t = \frac{\mu}{1-t}\pi_t(1-\pi_t)d\bar{B}_t, \quad \pi_0 = \pi.$$

- The optimal stopping problem to solve

$$V(t,\pi) = \inf_{0 \leqslant \tau \leqslant 1-t} \mathsf{E}_{t,\pi} G(t+\tau, \pi_{t+\tau}),$$

with the function $G(t,\pi) = ct + a\pi \wedge b(1-\pi)\,[= ct + G(\pi)]$ for all $(t,\pi) \in [0,1] \times [0,1]$. Under measure $\mathsf{P}_{t,\pi}(\pi_t = \pi) = 1$ the process $(\pi_{t+s})_{0 \leqslant s \leqslant 1-t}$ is the solution to the SDE

$$d\pi_{t+s} = \frac{\mu}{1-t-s}\,\pi_{t+s}(1-\pi_{t+s})d\bar{B}_{t+s}, \quad \pi_t = \pi,$$

- Remark: since the function G is *bounded and continuous* on $[0,1] \times [0,1]$, it follows from the general theory that the optimal stopping time in the considered problem exists.

Step 2: change of time and change of space

Method: let us assume that some process Y satisfies the SDE

$$dY_t = b(t, Y_t)dt + \sigma(t, Y_t)dB_t, \quad Y_0 = y_0.$$

- Assumptions: $T(t) \in C^1$, $T(t) \nearrow$, $\Psi(t, y) \in C^{1,2}$ and $\frac{\partial \Psi(t,y)}{\partial y} > 0$
- Change of time and change of space:

$$\begin{aligned} \eta &= \Psi(t, y) \\ \nu &= T(t) \end{aligned}$$

- Notations: $a(t, y) = \sigma^2(t, y)$,

$$B(\nu, \eta) = \frac{\frac{1}{2}a(t, y)\frac{\partial^2 \Psi(t,y)}{\partial y^2} + b(t, y)\frac{\partial \Psi(t,y)}{\partial y} + \frac{\partial \Psi(t,y)}{\partial t}}{\frac{\partial T(t)}{\partial t}},$$

$$A(\nu, \eta) = \frac{a(t, y)\left(\frac{\partial \Psi(t,y)}{\partial y}\right)^2}{\frac{\partial T(t)}{\partial t}}$$

- Trasformation:

$$\widehat{Y}_\nu = \Psi(\widehat{T}(\nu), Y_{\widehat{T}(\nu)}),$$
$$\widehat{T}(\nu) = \inf\{t \colon T(t) = \nu\}$$

So, Y with characteristics $(b, \sigma) \rightsquigarrow \widehat{Y}$ with characteristics (B, Σ) s.t.

$$d\widehat{Y}_\nu = B(\nu, \widehat{Y}_\nu)d\nu + \Sigma(\nu, \widehat{Y}_\nu)d\widehat{B}_\nu,$$

where $\Sigma^2 = A$ and $\widehat{B} = (\widehat{B}_\nu)_{\nu \geqslant 0}$ is a Wiener process.

Application

$$\eta = y, \quad \nu = \frac{t}{1-t}, \quad (t, y) \in [0, 1) \times [0, 1]$$
$$y = \eta, \quad t = \frac{\nu}{1+\nu}, \quad (\nu, \eta) \in \mathbb{R}_+ \times [0, 1]$$

$$\widehat{\pi}_\nu = \pi_{t(\nu)} = \pi_{\frac{\nu}{1+\nu}}, \quad \nu \in \mathbb{R}_+,$$

$$d\widehat{\pi}_\nu = \mu\widehat{\pi}_\nu(1 - \widehat{\pi}_\nu)d\widehat{\overline{B}}_\nu, \quad \widehat{\pi}_0 = \widehat{\pi}, \quad \nu \in \mathbb{R}_+$$

Transformed optimal stopping problem

$$V(\nu,\widehat{\pi}) = \inf_{\sigma \geqslant 0} \mathsf{E}_{\nu,\widehat{\pi}} \widehat{G}(\nu+\sigma, \widehat{\pi}_{\nu+\sigma}),$$

$$\widehat{G}(\nu,\widehat{\pi}) = \frac{c\nu}{1+\nu} + a\widehat{\pi} \wedge b(1-\widehat{\pi}) \left(= \frac{c\nu}{1+\nu} + G(\widehat{\pi}) \right).$$

where $\mathsf{P}_{\nu,\widehat{\pi}}(\widehat{\pi}_\nu = \widehat{\pi}) = 1$ and infimum is taken over all stopping times σ adapted to the natural filtration generated by the process $(\widehat{\pi}_{\nu+\zeta})_{\zeta \geqslant 0}$.

- Lagrange-Mayer form:

$$V(\nu,\widehat{\pi}) = \inf_{\sigma \geqslant 0} \mathsf{E}_{\nu,\widehat{\pi}} \left(c \int_0^\sigma \frac{d\zeta}{(1+\nu+\zeta)^2} + \widehat{G}(\nu, \widehat{\pi}_{\nu+\sigma}) \right)$$

  for $(\nu,\widehat{\pi}) \in \mathbb{R}_+ \times [0,1]$

- Note: one-to-one correspondence of Kolmogorov's time-space change guarantees the coincidence of the filtrations $\mathscr{F}_t^\pi \equiv \mathscr{F}_{\nu(t)}^{\widehat{\pi}}$

Step 3: structure of the optimal stopping time

One can find that:

- it is never optimal to stop on the line $\mathbb{R}_+ \times \{r\}$ with $r = \arg\max G(\pi)$
- the map $\pi \to V(t,\pi)$ is concave for every $t \geqslant 0$
- there exists boundaries $g_0(t) < r < g_1(t)$ for $t \in \mathbb{R}_+$ s.t.
    - ◇ *the continuation set* is given by

$$C = \{(t,\pi) \in \mathbb{R}_+ \times [0,1] \colon \pi \in (g_0(t), g_1(t))\}$$

    - ◇ *the stopping set* is the closure of the set

$$D = \{(t,\pi) \in \mathbb{R}_+ \times [0,1] \colon \pi \in [0, g_0(t)) \cup (g_1(t), 1]\}$$

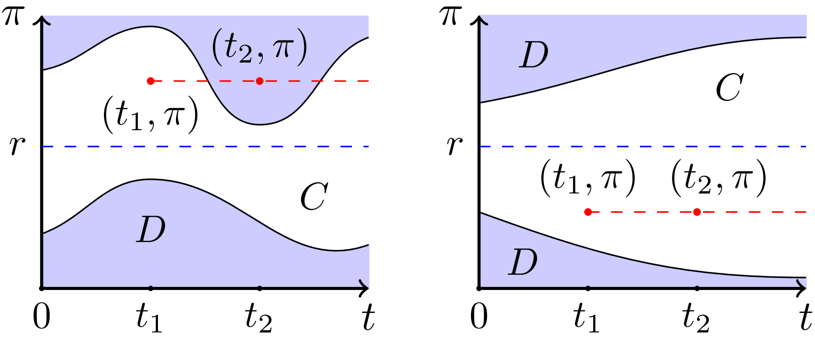
- the continuation set C is open and the stopping set D is closed
- $g_0(1) = 0$ and $g_1(1) = 1$

Remark: $V(t,\pi)$ has to be lsc (in fact, V is continuous!), whereas $G(t,\pi)$ has to be usc

Step 4: monotonicity of the boundaries

It follows from the fact that for arbirary numbers $t_2 > t_1$ on $\mathbb{R}_+$

$$V(t_2, \pi) - G(t_2, \pi) \leqslant V(t_1, \pi) - G(t_1, \pi)$$



Denote through $\sigma(t_1,\pi)$ and $\sigma(t_2,\pi)$ the optimal stopping moments in the correspondent problems. Then we have the following chain of inequalities:

$$V(t_2,\pi) - G(t_2,\pi) = \mathsf{E}\left(c \int\limits_0^{\sigma(t_2,\pi)} \frac{ds}{(1+t_2+s)^2} + G(\pi_{t_2+\sigma(t_2,\pi)}) - G(\pi) \right)$$

$$\leqslant \mathsf{E}\left(c \int\limits_0^{\sigma(t_1,\pi)} \frac{ds}{(1+t_2+s)^2} + G(\pi_{t_2+\sigma(t_1,\pi)}) - G(\pi) \right) \leqslant$$

$$\mathsf{E}\left(c \int\limits_0^{\sigma(t_1,\pi)} \frac{ds}{(1+t_1+s)^2} + G(\pi_{t_1+\sigma(t_1,\pi)}) - G(\pi) \right) = V(t_1,\pi) - G(t_1,\pi)$$

Hence, if the point $(t',\pi) \in C$ for some π then it is true that $V(t,\pi) - G(t,\pi) \leqslant V(t',\pi) - G(t',\pi) < 0$ for all $t \geqslant t' \Rightarrow (t,\pi) \in C$.

Step 5: continuity of $V(t,\pi)$

For the continuity of $V(t,\pi)$ as a mapping $(t,\pi) \to V(t,\pi)$ on $\mathbb{R}_+ \times [0,1]$ to be proved it is enough to verify

$$\pi \to V(t_0,\pi) \quad \text{is continuos in } \pi_0$$
$$t \to V(t,\pi) \quad \text{is continuous in } t_0 \text{ uniformly over } \pi \in [\pi_0-\delta, \pi_0+\delta]$$

$\pi \to V(t,\pi)$ is concave $\rightsquigarrow$ the first assertion holds

For the second one we have the following

$$0 \leqslant V(t_2, \pi) - V(t_1, \pi) \leqslant$$

$$\mathsf{E}_\pi \left[\int\limits_0^{\sigma(t_1,\pi)} \frac{ds}{(1+t_2+s)^2} + G(t_2, \pi_{t_2+\sigma(t_1,\pi)}) \right] -$$

$$\mathsf{E}_\pi \left[\int\limits_0^{\sigma(t_1,\pi)} \frac{ds}{(1+t_1+s)^2} + G(t_1, \pi_{t_1+\sigma(t_1,\pi)}) \right] =$$

$$\mathsf{E}_\pi \int\limits_0^{\sigma(t_1,\pi)} \left(\frac{1}{(1+t_2+s)^2} - \frac{1}{(1+t_1+s)^2} \right) ds + \left(\frac{ct_2}{1+t_2} - \frac{ct_1}{1+t_1} \right) \to 0$$

as $t_2 \downarrow t_1$.

Step 6: smooth fit principle

$$Smooth fit$$

$$\Updownarrow$$

$$\pi \to V(t,\pi) \quad \text{belongs to} \quad C^1 \quad \text{on the boundaries} \quad (g_0, g_1)$$

- For any point $(t,\pi) \in \mathbb{R}_+ \times (0,1)$ such that $\pi = g_0(t)$, and for all $\varepsilon > 0$ with $\pi < \pi + \varepsilon < r$ we have

$$\frac{V(t,\pi+\varepsilon) - V(t,\pi)}{\varepsilon} \leqslant \frac{G(t,\pi+\varepsilon) - G(t,\pi)}{\varepsilon}$$

$$\Downarrow \varepsilon \downarrow 0$$

$$\frac{\partial^+ V}{\partial \pi}(t,\pi) \leqslant \frac{\partial G}{\partial \pi}(t,\pi)$$

- The reverse inequality holds (but the proof is much trickier)

$$\frac{\partial^+ V}{\partial \pi}(t,\pi) \geqslant \frac{\partial G}{\partial \pi}(t,\pi)$$

Step 7: restraints on the optimal boundaries

Let us introduce two time-homogeneous optimal stopping problems

$$H(\nu, \widehat{\pi}; c) = \inf_{\sigma \geqslant 0} \mathsf{E}_{\widehat{\pi}} \left(c \int_0^{\sigma} 1 \, d\zeta + G(\nu, \widehat{\pi}_\sigma) \right)$$

$$K(\nu, \widehat{\pi}; c) = \inf_{\sigma \geqslant 0} \mathsf{E}_{\widehat{\pi}} \left(c \int_0^{\sigma} e^{-\zeta} d\zeta + e^{-\sigma} G(\nu, \widehat{\pi}_\sigma) \right).$$

One can easily check validity of the following relations

$$K\left(\cancel{\nu}, \widehat{\pi}; c \cdot e^{-(1+\nu)}\right) \leqslant V(\nu, \widehat{\pi}) \leqslant H\left(\cancel{\nu}, \widehat{\pi}; c \cdot (1+\nu)^{-2}\right)$$

General theory says that $C = \{(\nu, \widehat{\pi}) \colon V(\nu, \widehat{\pi}) < G(\nu, \widehat{\pi})\}$ and $D = \{(\nu, \widehat{\pi}) \colon V(\nu, \widehat{\pi}) = G(\nu, \widehat{\pi})\}$. Hence, we conclude that

$$m_0^K(\nu) \leqslant \widehat{g}_0(\nu) \leqslant M_0^H(\nu),$$
$$M_1^H(\nu) \leqslant \widehat{g}_1(\nu) \leqslant m_1^K(\nu),$$

where $\nu(t) = t(1-t)^{-1}$ for all $t \in [0, 1)$.

Step 8: continuity of the boundaries (g_0, g_1)

- Continuity of $g_i(t)$ from the left

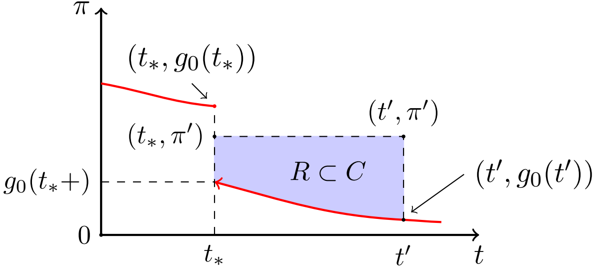
Let us fix some t and concider a sequence of $t_n \uparrow t$ as $n \to \infty$. The limit $\lim_{n\to\infty} g_i(t_n)$ exists (we invoke the monotonicity argument here) and equals $g_i(t-)$. Since the points $(t_n, g_i(t_n)) \in \bar{D}$ for all $n \geqslant 1$ and $\bar{D}$ is a close set we have that $(t, g_i(t-)) \in \bar{D}$, as well. However, using the structer of the set D it is easy to see that $g_0(t) \geqslant g_0(t-)$ and $g_1(t) \leqslant g_1(t-)$. The reverse inequalities are obvious due to increasing and decreasing of the boundaries $g_i(t)$. Thus, we have

$$g_i(t-) = g_i(t) \quad \text{for all} \quad t > 0$$

Remainder:

$$D = \{(t, \pi) \colon \pi \in [0, g_0(t)) \cup (g_1(t), 1]\}$$

- Continuity from the right



$$V(t,\pi)|_{\pi=g_0(t)+} = \frac{t}{1+t} + a g_0(t), \quad V(t,\pi)|_{\pi=g_1(t)-} = \frac{t}{1+t} + b(1-g_1(t))$$

$$\left.\frac{\partial V}{\partial \pi}\right|_{\pi=g_0(t)+} = a, \quad \left.\frac{\partial V}{\partial \pi}\right|_{\pi=g_1(t)-} = -b$$

These allow us to apply the Newton-Leibniz formula for all $(t,\pi) \in R$

$$V(t,\pi) - G(t,\pi) = \int_{g_0(t)}^{\pi} \int_{g_0(t)}^{u} \left(\frac{\partial^2 V}{\partial \pi^2} - \frac{\partial^2 G}{\partial \pi^2} \right)(t,v)\,dv\,du$$

It can be shown that

- $\frac{\partial V}{\partial t}(t,\pi) > 0$ for all $(t,\pi) \in C$
- the function $V(t,\pi)$ solves the equation

$$L_\pi V(t,\pi) = 0 \quad \text{for all} \quad (t,\pi) \in C$$

$$\Downarrow$$

For all $t_* < t \leqslant t'$, $g_0(t') < \pi \leqslant \pi'$ with δ being small enough, $s > 0$

$$\frac{\partial^2 V}{\partial \pi^2}(t,\pi) = -\frac{2}{\mu^2}\frac{1}{\pi^2(1-\pi)^2}\frac{\partial V}{\partial t}(t,\pi) \leqslant -\frac{2\delta}{\mu^2}\frac{c}{(1+t+s)^2}$$

$$V(t',\pi') - G(t',\pi') \leqslant -\frac{2\delta}{\mu^2}\frac{c}{(1+t'+s)^2}\frac{(\pi' - g_0(t'))^2}{2}$$

$$\Downarrow t' \downarrow t_*$$

$$V(t_*,\pi') - G(t_*,\pi') \leqslant -\frac{2\delta}{\mu^2}\frac{c}{(1+t_*+s)^2}\frac{(\pi' - g_0(t_*+))^2}{2} < 0$$

Step 9: free-boundary problem

Summarizing the facts proved, we have:

$$\tau_* = \inf\{0 \leqslant s \leqslant 1 - t \colon \pi_{t+s} \notin (g_0(t+s), g_1(t+s))\}$$

(the infimum of an empty set is supposed to be equal to $1 - t$), where the pair (g_0, g_1) has the following properties:

$$\begin{aligned}
g_0 \colon [0,1] \to [0,1] \quad & \text{is continuous and decreasing} \\
g_1 \colon [0,1] \to [0,1] \quad & \text{is continuous and increasing} \\
m_0\left(\frac{t}{1-t}\right) \leqslant g_0(t) \leqslant M_0\left(\frac{t}{1-t}\right) < r \quad & \text{for all} \quad 0 \leqslant t < 1 \\
r < M_1\left(\frac{t}{1-t}\right) \leqslant g_1(t) \leqslant m_1\left(\frac{t}{1-t}\right) \quad & \text{for all} \quad 0 \leqslant t < 1 \\
g_0(1) = 0,\ g_1(1) = 1 \quad &
\end{aligned}$$

$m_i(\nu)$ and $M_i(\nu)$, $i = 0, 1$, being the solutions of two different systems of transcendental equations.

The infinitesimal operator L of the process (t, π_t) is defined as:

$$(Lf)(t,\pi) = \left(\frac{\partial f}{\partial t} + \frac{\mu^2}{2}\frac{\pi^2(1-\pi)^2}{(1-t)^2}\frac{\partial^2 f}{\partial \pi^2}\right)(t,\pi), \quad f \in C^{1,2}([0,1)\times[0,1]$$

- The free-boundary problem:

$$\begin{aligned}
(LV)(t,\pi) = 0, &\quad \text{for} \quad (t,\pi) \in C \\
V(t,\pi)|_{\pi=g_0(t)+} = ct + ag_0(t), &\quad V(t,\pi)|_{\pi=g_1(t)-} = ct + b(1-g_1(t)) \\
\frac{\partial V}{\partial \pi}(t,\pi)|_{\pi=g_0(t)+} = a, &\quad \frac{\partial V}{\partial \pi}(t,\pi)|_{\pi=g_1(t)-} = -b \\
V(t,\pi) < G(t,\pi) &\quad \text{for} \quad (t,\pi) \in C \\
V(t,\pi) = G(t,\pi) &\quad \text{for} \quad (t,\pi) \in D,
\end{aligned}$$

with $G(t,\pi) = ct + a\pi \wedge b(1-\pi)$ and the sets C and D given by

$$C = \inf\{(t,\pi) \in [0,1) \times [0,1] \colon \pi \in (g_0(t), g_1(t))\},$$
$$D = \inf\{(t,\pi) \in [0,1) \times [0,1] \colon \pi \in [0, g_0(t)) \cup (g_1(t), 1]\},$$

- By Peskir's change-of-variable formula with local time on curves:

$$V(t+s, \pi_{t+s}) = V(t, \pi_t) +$$

$$\int\limits_0^s (LV)\,(t+u, \pi_{t+u}) \mathbb{1}(\pi_{t+u} \neq g_0(t+u), \pi_{t+u} \neq g_1(t+u))du +$$

$$\sum_{i=0}^{1} \frac{1}{2} \int\limits_0^s \left(V_t(t+u, \pi_{t+u}+) + V_t(t+u, \pi_{t+u}-)\right) \mathbb{1}(\pi_{t+u} = g_i(t+u))du +$$

$$\sum_{i=0}^{1} \frac{1}{2} \int\limits_0^s \left(V_\pi(t+u, \pi_{t+u}+) + V_\pi(t+u, \pi_{t+u}-)\right) \mathbb{1}(\pi_{t+u} = g_i(t+u))d\pi_{t+u}$$

$$\sum_{i=0}^{1} \frac{1}{2} \int\limits_0^s \Delta V_\pi(t+u, \pi_{t+u}) \mathbb{1}(\pi_{t+u} = g_i(t+u))dL_u^{g_i(\cdot)} + M_s,$$

$$M_s = \int_0^s V_\pi(t+u, \pi_{t+u}) \mathbb{1}(\pi_{t+u} \neq g_0(t+u), \pi_{t+u} \neq g_1(t+u))d\pi_{t+u}$$

Taking the expectation w.r.t. the measure $\mathsf{P}_{t,\pi}$, we finally obtain

$$\mathsf{E}_{t,\pi} V(t+s, \pi_{t+s}) = V(t,\pi) + \\ c \int_0^s \mathsf{P}_{t,\pi}(\pi_{t+u} < g_0(t+u))du + c \int_0^s \mathsf{P}_{t,\pi}(\pi_{t+u} > g_1(t+u))du$$

for all $s \in [0, 1-t]$.

Setting $\pi = g_i(t)$, $i = 0, 1$, and $s = 1-t$, we receive the required system of nonlinear integral equations ($i = 0, 1$)

$$c \sum_{j=0}^{1} (-1)^{j+1} \int\limits_0^{1-t} \mathsf{P}_{t, g_i(t)} \left(\pi_{t+u}^{g_i(t)} \leqslant g_j(t+u) \right) du = a g_i(t) \wedge b(1 - g_i(t))$$

Step 10: uniqueness

The uniqueness of the solution in the class of functions we are dealing with is pretty the same as in the work of Gapeev and Peskir, 2003. $\square$

6. Conclusion

- Problem:

$$X_t = \mu\theta t + B_t^0, \quad \mu \neq 0$$

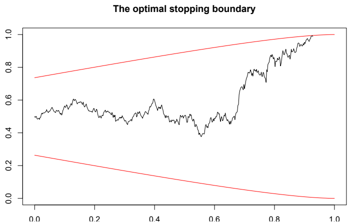
$$H_0\colon \theta = 0 \qquad H_1\colon \theta = 1$$

$$V(\pi) = \inf_{(\tau,d)} \mathsf{E}_\pi \left[c\tau + a\mathbb{1}(d=0,\theta=1) + b\mathbb{1}(d=1,\theta=0)\right],$$

- The optimal decision rule:

$$\tau^* = \inf\{0 \leqslant t \leqslant 1\colon \pi_t \notin (g_0(t), g_1(t))\}$$

$$d^* = \begin{cases} 1 \ (\text{accept } H_1), \ \text{if } \pi_{\tau^*} = g_1(\tau^*) \\ 0 \ (\text{accept } H_0), \ \text{if } \pi_{\tau^*} = g_0(\tau^*) \end{cases}$$


The optimal stopping boundary
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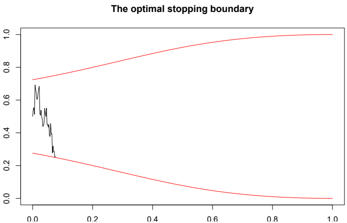
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0


The optimal stopping boundary
0.0
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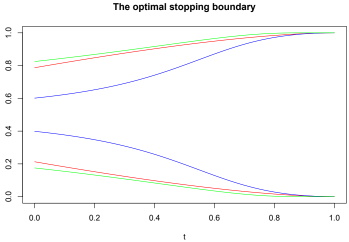
The optimal stopping boundary
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
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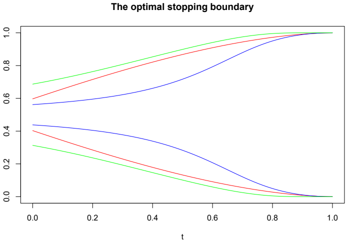
0.6
0.8
1.0
t


The optimal stopping boundary
0.0
0.2
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0.6
0.8
1.0
0.0
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0.4
0.6
0.8
1.0
t

Thank you for your attention!

Step 6: smooth fit principle

In this section we would like to convinced that $\pi \to V(t,\pi)$ belongs to C^1 on the boundaries (g_0, g_1).

Let us fix some point $(t,\pi) \in \mathbb{R}_+ \times (0,1)$ such that $\pi = g_0(t)$. Then for all $\varepsilon > 0$ with $\pi < \pi + \varepsilon < r$ we have

$$\frac{V(t,\pi+\varepsilon) - V(t,\pi)}{\varepsilon} \leqslant \frac{G(t,\pi+\varepsilon) - G(t,\pi)}{\varepsilon}.$$

Thus, taking the limit as $\varepsilon \downarrow 0$ we get

$$\frac{\partial^+ V}{\partial \pi}(t,\pi) \leqslant \frac{\partial G}{\partial \pi}(t,\pi).$$

The only thing to do is to set the reverse inequality. The idea is to invoke subharmonic characterisation of the function $V(t,\pi)$ and the scale and measure functions for diffusion process π_t.

In doing so, let us introduce a stopping moment

$$\tau_\varepsilon = \inf\{s \geqslant 0 \colon \pi_s \notin (\pi - \varepsilon, \pi + \varepsilon)\}$$

Then

$$\begin{aligned}\mathsf{E}_{t,\pi} V(t, \pi_{t+\tau_\varepsilon}) = &\mathsf{P}_{t,\pi}(\pi_{t+\tau_\varepsilon} = \pi + \varepsilon) V(t, \pi + \varepsilon) + \\ &\mathsf{P}_{t,\pi}(\pi_{t+\tau_\varepsilon} = \pi - \varepsilon) V(t, \pi - \varepsilon)\end{aligned}$$

Note additionally that two following relations are valid

$$\begin{aligned}\mathsf{E}_{t,\pi} V(t, \pi_{t+\tau_\varepsilon}) &\geqslant \mathsf{E}_{t,\pi} V(t + \tau_\varepsilon, \pi_{t+\tau_\varepsilon}) + \mathsf{E}_{t,\pi}\left[G(t, \pi_{t+\tau_\varepsilon}) - G(t + \tau_\varepsilon, \pi_{t+\tau_\varepsilon})\right] \\ &\geqslant V(t, \pi) + \mathsf{E}_{t,\pi}\left(\frac{ct}{1+t} - \frac{c(t+\tau_\varepsilon)}{1+t+\tau_\varepsilon}\right),\end{aligned}$$

where we have used the fact prooved that $V(t, \pi) - G(t, \pi) \leqslant V(t', \pi) - G(t', \pi)$ for all $t \geqslant t'$

$$V(t, \pi) = \mathsf{P}_\pi(\pi_{\tau_\varepsilon} = \pi + \varepsilon) \cdot V(t, \pi) + \mathsf{P}_\pi(\pi_{\tau_\varepsilon} = \pi - \varepsilon) \cdot G(t, \pi)$$

Combining those results we may conclude that

$$\frac{V(t,\pi+\varepsilon)-V(t,\pi)}{\varepsilon} \geqslant \frac{G(t,\pi)-G(t,\pi-\varepsilon)}{\varepsilon}\cdot\frac{\mathsf{P}(\pi_{\tau_\varepsilon}=\pi-\varepsilon)}{\mathsf{P}(\pi_{\tau_\varepsilon}=\pi+\varepsilon)}+$$
$$\frac{1}{\varepsilon\,\mathsf{P}_\pi(\pi_{\tau_\varepsilon}=\pi+\varepsilon)}\cdot\mathsf{E}_{t,\pi}\left(\frac{ct}{1+t}-\frac{c(t+\tau_\varepsilon)}{1+t+\tau_\varepsilon}\right)$$

Well known fact: $\mathsf{P}(\pi_{\tau_\varepsilon}=\pi+\varepsilon))=(S(\pi)-S(\pi-\varepsilon)/(S(\pi+\varepsilon)-S(\pi-\varepsilon))$ and $\mathsf{P}(\pi_{\tau_\varepsilon}=\pi-\varepsilon))=(S(\pi+\varepsilon)-S(\pi)/(S(\pi+\varepsilon)-S(\pi-\varepsilon))$, $S(x)=x$ for $x\in[0,1]$ being the scale functionof the process π_t.

Moreover one can get the following estimation

$$\mathsf{E}_\pi\left[\tau_\varepsilon\right]=\int\limits_{\pi-\varepsilon}^{\pi+\varepsilon}G_{\pi-\varepsilon,\pi+\varepsilon}(\pi,y)m(dy)\leqslant K\cdot\varepsilon^2$$

for some K large enough (not depending on ε).

We have used Green function

$$G_{\pi_0,\pi_1}(x,y) = \begin{cases} (\pi_1 - x)(y - \pi_0)/(\pi_1 - \pi_0) & \text{if} \quad \pi_0 \leqslant y \leqslant x, \\ (\pi_1 - y)(x - \pi_0)/(\pi_1 - \pi_0) & \text{if} \quad x \leqslant y \leqslant \pi_1, \end{cases}$$

for any $[\pi_0, \pi_1] \subset (0,1)$.

Whereas the measure function $m(dy)$ is given by $m(dy) = 2[\mu y(1 - y)]^{-1} dy$ for all $y \in (0,1)$.

Since functions S and G are differentiable in $\pi = g_0(t)$ for all $0 \leqslant \pi < r$ we have

$$\frac{\partial^+ V}{\partial \pi}(t,\pi) \geqslant \frac{\partial G}{\partial \pi}(t,\pi) \cdot \frac{S'(\pi)}{S'(\pi)} = \frac{\partial G}{\partial \pi}(t,\pi).$$

as $\varepsilon \downarrow 0$

The proof for the boundary g_1 is quite the same with trivial changing.