Ruin probabilities with investments in a risky asset with the price given by a geometric Lévy process

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Abstract

- We consider a model of describing the evolution of the capital of a venture company selling innovations when it invests its reserve into a risky asset with the price given by a geometric Lévy process.
- We find the **exact** asymptotic of the ruin probabilities. Under some natural conditions it decays as a power function. The rate of decay is a positive root of equation determined by characteristics of the price process. When the price follows a gBm the results are reduced to those of our previous work where we used the method of differential equations assuming exponentially distributed jumps.
- Our proofs are based on the asymptotic theory for renewal equations, in particular on a very recent result by Guivarc'h and Le Page.

Exit probabilities for linear equations

- $R=(R_t)_{t\geq 0}$, $P=(P_t)_{t\geq 0}$ are independent Lévy processes, $P_0=R_0=0$, $\Delta R>-1$.
- $X=X^u$ describes the capital evolution ; it is given by the linear equation $dX=X_-dR+dP$, $X_0=u$, i.e.

$$X_t=u+X_-\cdot R_t+P_t=u+\int_{]0,t]}X_{s-}dR_s+P_t, \qquad u\in\mathbb{R}_+.$$

- Since $[P,R]=0$ we have the analog of the Cauchy formula

$$X=\mathcal{E}(R)(u+\mathcal{E}_-^{-1}(R)\cdot P)=\mathcal{E}(R)(u-Y),$$

where the Doléans exponential $\mathcal{E}(R)$ is the price process, $\mathcal{E}(R):=1+\mathcal{E}_-(R)\cdot R$ and $Y:=-\mathcal{E}_-^{-1}(R)\cdot P$.
- Ruin time : $\tau^u:=\inf\{t:\ X_t^u\leq 0\}$, the exit time from $]0,\infty[$.
- Ruin probability : $\Psi(u):=\mathbf{P}(\tau^u<\infty)$.

Lévy triplets

- (a,σ^2,Π), (a_P,σ_P^2,Π_P) are the Lévy triplets of R, P, i.e.

$$R_t = at + \sigma W_t + h * (\mu - \nu)_t + \bar{h} * \mu_t,$$

compensator $\nu(dt,dx) = dt\Pi(dx)$ with $\Pi(|x|^2 \wedge 1) < \infty$,
truncation function $h(x) := xI_{\{|x|\le 1\}}$, $\bar{h}(x) := xI_{\{|x|>1\}}$.
The process P has a similar representation with W^P, μ^P, ν^P.

To exclude trivial or known cases we work under the following

- **Assumption** $\Pi(]-\infty,-1]$; *σ^2 and Π do not vanish simultaneously, P is not a subordinator.*

- Stochastic exponential can be written as the usual one :

$$X_t^u = e^{V_t}(u - Y_t),$$

$$Y_t := -\int_{]0,t]} \mathcal{E}_{s-}^{-1}(R)dP_s = -\int_{]0,t]} e^{-V_{s-}} dP_s.$$

Main result

The Lévy process V (the log price) has the form

$$V_t = at - (1/2)\sigma^2 t + \sigma W_t + h * (\mu - \nu)_t + (\ln(1+x) - h) * \mu_t.$$

and has the triplet $(a_V, \sigma^2, \Pi\varphi^{-1})$ where $\varphi : x \mapsto \ln(1+x)$,

$$a_V = a - (1/2)\sigma^2 + \Pi(h(\ln(1+x)) - h).$$

For r.v. V_1 the cumulant generating function

$$H(q) := \ln \mathbf{E}\, e^{-qV_1} = -a_V q + \frac{\sigma^2}{2} q^2 + \Pi\big(e^{-q\ln(1+x)} - 1 + qh(\ln(1+x))\big).$$

Put $\underline{q} := \inf\{q \le 0 \colon \; H(q) < \infty\}$, $\bar{q} := \sup\{q \ge 0 \colon \; H(q) < \infty\}$.

Theorem

Suppose that H has a root $\beta > 0$ laying in $\operatorname{int}\operatorname{dom} H$ and $\Pi(|\bar{h}|^\beta) < \infty$. If the law $\mathcal{L}(V_T)$ is non-arithmetic for some $T > 0$, then $\Psi(u) = O(u^{-\beta})$ as $u \to \infty$.
If, moreover, $\Pi_P(]-\infty, 0[) = 0$, then $\Psi(u) \sim C_\infty u^{-\beta}$ for $C_\infty > 0$.

Reduction

So, $X^u_t = e^{V_t}(u - Y_t)$. Obviously, $\tau^u = \inf\{t \geq 0 : \ Y_t \geq u\}$. Put $G(u) := \mathbf{P}(Y_\infty > u)$.

Lemma

If $Y_t \to Y_\infty$ a.s. where Y_∞ is finite and unbounded from above, then for all $u > 0$

$$G(u) \leq \Psi(u) = \frac{G(u)}{\mathbf{E}\left(G(X_{\tau^u}) \,|\, \tau^u < \infty\right)} \leq \frac{G(u)}{G(0)}. \tag{1}$$

In particular, if $\Pi_P(]-\infty, 0]) = 0$, then $\Psi(u) = G(u)/G(0)$.

Proof. Let τ be a stopping time, ξ be a $\mathcal{F}_\tau$-measurable r.v.,

$$Y_{\tau,\infty} := \begin{cases} -\lim_{N\to\infty} \int_{]\tau,\tau+N]} e^{-(V_{t-}-V_\tau)} dP_t, & \tau < \infty, \\ 0, & \tau = \infty. \end{cases}$$

is well defined.

On the set $\{\tau < \infty\}$

$$Y_{\tau,\infty} = e^{V_\tau}(Y_\infty - Y_\tau) = X_\tau + e^{V_\tau}(Y_\infty - u).$$

Since the Lévy process Y starts afresh at τ, the conditional distribution of $Y_{\tau,\infty}$ given $(\tau,\xi) = (t,x) \in \mathbb{R}_+ \times \mathbb{R}$ is the same as $\mathcal{L}(Y_\infty)$. It follows that

$$\mathbf{P}\left(Y_{\tau,\infty} > \xi,\ \tau < \infty\right) = \mathbf{E}\, G(\xi)\, I_{\{\tau<\infty\}}.$$

Thus, if $\mathbf{P}(\tau < \infty) > 0$, then

$$\mathbf{P}\left(Y_{\tau,\infty} > \xi,\ \tau < \infty\right) = \mathbf{E}\left(G(\xi)\,|\,\tau < \infty\right)\, \mathbf{P}(\tau < \infty)\,.$$

Noting that $\Psi(u) := \mathbf{P}(\tau^u < \infty) \geq \mathbf{P}(Y_\infty > u) > 0$, we get that

$$\begin{aligned} G(u) &= \mathbf{P}\left(Y_\infty > u,\ \tau^u < \infty\right) = \mathbf{P}\left(Y_{\tau^u,\infty} > X_{\tau^u},\ \tau^u < \infty\right) \\ &= \mathbf{E}\left(G(X_{\tau^u})\,|\,\tau^u < \infty\right)\, \mathbf{P}(\tau^u < \infty). \end{aligned}$$

The result follows since $X_{\tau^u} \leq 0$ on the set $\{\tau^u < \infty\}$. In the case where $\Pi_P(]-\infty,0]) = 0$, the process X^u crosses zero in a continuous way, i.e. $X_{\tau^u} = 0$ on this set.

Reduction to distributional equations

Put

$$M_n := e^{-(V_n - V_{n-1})}, \qquad Q_n := -\int_{]n-1,n]} e^{-(V_t - V_{n-1})} dP_t.$$

Clearly, $\mathcal{L}(M_1, Q_1) = \mathcal{L}(M_n, Q_n)$ and

$$Y_n = -e^{-V_{n-1}} \int_{]0,n]} e^{-(V_t - V_{n-1})} dP_t = Q_1 + M_1 Q_2 + M_1 M_2 Q_3 + \dots$$

Thus, $Y_n = Q_1 + M_1 Y_{n-1,1}$ where

$$Y_{n-1,1} := Q_2 + M_2 Q_3 + \dots M_2 \dots M_{n-1} Q_n.$$

Suppose that $Y_n \to Y_\infty$. Then $Y_\infty = Q_1 + M_1 Y_{\infty,1}$ a.s. Hence,

$$Y_\infty \overset{d}{=} Q + M\, Y_\infty\,, \quad Y_\infty \text{ independent of } (M, Q).$$

Implicit renewal theory

Let $M > 0$ be such that $\mathcal{L}(\ln M)$ is non-arithmetic and

$$\mathbf{E}\, M^{\beta} = 1\,, \quad \mathbf{E}\, M^{\beta}\, (\ln M)^{+} < \infty \qquad \text{for some } \beta > 0.$$

Then $\ln \mathbf{E} M \in [-\infty, 0[$ and $\kappa := \mathbf{E} M^{\beta} \ln M \in]0, \infty[$.

Lemma (Goldie, 1991)

Let M satisfies the conditions above, $\mathbf{E}\,|Q|^{\beta} < \infty$. Then the distributional equation $Y_{\infty} \overset{d}{=} Q + M\, Y_{\infty}$, Y_{∞} independent of (M, Q) has a unique solution Y_{∞} and

$$\lim_{u \to \infty} u^{\beta}\, \mathbf{P}(Y_{\infty} > u) = C_{+} := \frac{1}{\beta\, \kappa}\, \mathbf{E}\, \left(((Q + MY_{\infty})^{+})^{\beta} - ((MY_{\infty})^{+})^{\beta} \right).$$

Lemma (Guivarc'h, Le Page, 2015 ; Buraczewski, Damek, 2016)

$$C_{+} > 0 \;\Leftrightarrow\; Y_{\infty} \;\; \textit{unbounded from above.}$$

Moments of the maximal function $Y_1^* := \sup_{t \le 1} |Y_t|$

Lemma

If $\Pi_P(|\bar{h}|^p) + \mathbf{E} \sup_{t \le 1} e^{-pV_t} < \infty$ for $p > 0$, then $\mathbf{E}\, Y_1^{*p} < \infty$.

The proof follows from the Novikov inequalities for the integral $I = g * (\mu^P - \nu^P)$ where $g^2 * \nu_1^P < \infty$. In dependence of the parameter $\alpha \in [1,2]$ they have the following form :

$$\mathbf{E} I_1^{*p} \le C_{p,\alpha} \begin{cases} \mathbf{E}\left(|g|^\alpha * \nu_1^P\right)^{p/\alpha}, & \forall\, p \in]0,\alpha], \\ \mathbf{E}\left(|g|^\alpha * \nu_1^P\right)^{p/\alpha} + \mathbf{E}\, |g|^p * \nu_1^P, & \forall\, p \in [\alpha, \infty[. \end{cases}$$

If $H(q) < \infty$, then the process $m_t(q) := e^{-qV_t - tH(q)}$ is a martingale and $\mathbf{E}\, e^{-qV_t} = e^{tH(q)}$, $t \in [0,1]$. From this it is easy to deduce that

$$\mathbf{E} \sup_{t \le 1} e^{-pV_t} < \infty \qquad \forall\, p \in]\underline{q}, \bar{q}[.$$

Convergence of Y

The convergence Y_t as $t \to \infty$ can be easily established under very weak assumptions.

Proposition

If there is $p > 0$ such that $H(p) < 0$, and $\Pi_P(|\bar{h}|^p) < \infty$, then Y_t converge a.s. to a finite r.v. Y_∞ unbounded from above and solving the distributional equation

$$Y_\infty \stackrel{d}{=} Y_1 + M_1 Y_\infty\,, \quad Y_\infty \ \text{ independent of } (M_1, Y_1).$$

Proof. We assume wlg that $p < 1$ and $H(p+) \neq \infty$. For $j \geq 2$

$$Y_j - Y_{j-1} = M_1 \ldots M_{j-1} Q_j, \qquad .$$

Since $\rho := \mathbf{E} M_1^p = e^{H(p)} < 1$ and $\mathbf{E} M_1 \ldots M_{j-1} |Q_j| = \rho^j \mathbf{E} |Y_1|^p$, we have $\mathbf{E} \sum_{j\geq 1} |Y_j - Y_{j-1}|^p < \infty$. Hence, $\sum_{j \geq 1} |Y_j - Y_{j-1}|^p < \infty$ a.s. But then $\sum_{j\geq 1} |Y_j - Y_{j-1}| < \infty$ a.s. and the sequence Y_n converges to some finite random variable Y_∞.

Put

$$\Delta_n := \sup_{n-1\le v\le n} \left| \int_{]n-1,v]} e^{-V_{s-}}\, dP_s \right|, \qquad n \ge 1.$$

Note that

$$\mathbf{E}\,\Delta_n^p = \mathbf{E} \prod_{j=1}^{n-1} M_j^p \sup_{n-1\le v\le n} \left| \int_{]n-1,v]} e^{-(V_{s-}-V_{n-1})}\, dP_s \right|^p = \rho^{n-1}\, \mathbf{E}\, Y_1^{*p} < \infty.$$

For any $\varepsilon > 0$ we get using the Chebyshev inequality that

$$\sum_{n\ge 1} \mathbf{P}(\Delta_n > \varepsilon) \le \varepsilon^{-p} \mathbf{E}\, Y_1^{*p} \sum_{n\ge 1} \rho^{n-1} < \infty.$$

By the Borel–Cantelli lemma $\Delta_n(\omega) \le \varepsilon$ for all $n \ge n_0(\omega)$ for each ω except a null-set. This implies the convergence $Y_t \to Y_\infty$ a.s.

Let us consider the sequence

$$Y_{1,n} := Q_2 + M_2 Q_3 + \cdots + M_2 \dots M_n Q_{n+1}$$

converging a.s. to a random variable $Y_{1,\infty}$ distributed as Y_∞. Passing to the limit in the obvious identity $Y_n = Q_1 + M_1 Y_{1,n-1}$ we obtain that $Y_\infty = Q_1 + M_1 Y_{1,\infty}$. For finite n the random variables $Y_{1,n}$ and (M_1, Q_1) are independent, $\mathcal{L}(Y_{1,n}) = \mathcal{L}(Y_n)$. Hence, $Y_{1,\infty}$ and (M_1, Q_1) are independent, $\mathcal{L}(Y_{1,\infty}) = \mathcal{L}(Y_\infty)$ and $\mathcal{L}(Y_\infty) = \mathcal{L}(Q_1 + M_1 Y_{1,\infty})$.

It remains to check that Y_∞ is unbounded from above.

Ruin with probability one

Proposition

Suppose that $\mathbf{E}M_1^{-\delta} < 1$ *and* $\mathbf{E}M_1^{-\delta}|Q_1|^{\delta} < \infty$ *for some* $\delta \in]0,1[$ *and* Q_1 *is unbounded from above. Then* $\Psi(u) \equiv 1$.

More specific conditions of the ruin almost surely in terms of triplets :

Theorem

Suppose that $0 \in \operatorname{int} \operatorname{dom} H$ *and* $\Pi_P(|\bar{h}|^{\varepsilon}) < \infty$ *for some* $\varepsilon > 0$. *If* $a_V + \Pi(\bar{h}(\ln(1+x))) \le 0$, *then* $\Psi(u) \equiv 1$.

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