Testing hypotheses about the drift of a Brownian motion

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Introduction

Suppose we observe the process

$$X_t = \mu t + B_t, \qquad t \geqslant 0,$$

where B is a Brownian motion, μ is an unknown drift coefficient.

We consider the problem of sequentially testing the hypotheses

$H_+ \colon \mu \geqslant 0$ and $H_- \colon \mu < 0$.

Outline

1. The Chernoff problem
2. Optimal stopping problems with unbounded pay-off functions
3. The Kiefer-Weiss problem
4. FBm case

The Chernoff problem

Assumption: $\mu \sim \mathcal{N}(0, \sigma_0^2)$ and is independent of B.

A decision rule is a pair (τ, d):

– τ is a stopping time of X

– d is an $\mathcal{F}_\tau$-measurable function, $d(\omega) \in \{-1, 1\}$

The Chernoff problem

$$\mathrm{E}\big[c\tau + k|\mu|\mathbf{I}(d \neq \mathrm{sgn}(\mu))\big] \xrightarrow[(\tau,d)]{} \min,$$

where $c, k > 0$ are constants; without loss of generality $c = k = 1$.

Known results

Chernoff and Breakwell: the optimal decision rule is

$$\tau^* = \inf\left\{t \geqslant 0 : |X_t| \geqslant b^*\left(t + \frac{1}{\sigma_0^2}\right)\right\}, \quad d^* = \operatorname{sgn}(X_{\tau^*})$$

where $b^*(t)$ is some function on $\mathbb{R}_+$.

They found its asymptotics for $t \to 0$, $t \to \infty$ ($\sigma_0 \to \infty$, $\sigma_0 \to 0$).

Solution of the Chernoff problem

Fix σ_0 and introduce the process $W = (W_t)_{0 \leqslant t \leqslant 1}$,

$$W_t = \sigma_0(1-t)X_{\frac{t}{\sigma_0^2(1-t)}},$$

which is a Brownian motion.

Theorem

The solution of the Chernoff problem is

$$\tau^* = \frac{\tau'^*}{\sigma_0^2(1-\tau'^*)}, \qquad d^* = \operatorname{sgn}(X_{\tau^*}).$$

where

$$\tau'^* = \inf\{0 \leqslant t \leqslant 1 : |W_t| \geqslant a^*_{\sigma_0}(t)\},$$

and $a^*_{\sigma_0}(t)\colon [0,1] \to \mathbb{R}_+$ is a non-increasing function which is the unique solution of the equation (with some concrete function H)

$$(1-t)H(1-t, a(t)) = \int_t^1 \tfrac{1}{\sigma_0^3(1-s)^2}\left[\Phi\left(\tfrac{a(s)-a(t)}{\sqrt{s-t}}\right) - \Phi\left(\tfrac{-a(s)-a(t)}{\sqrt{s-t}}\right)\right] ds$$

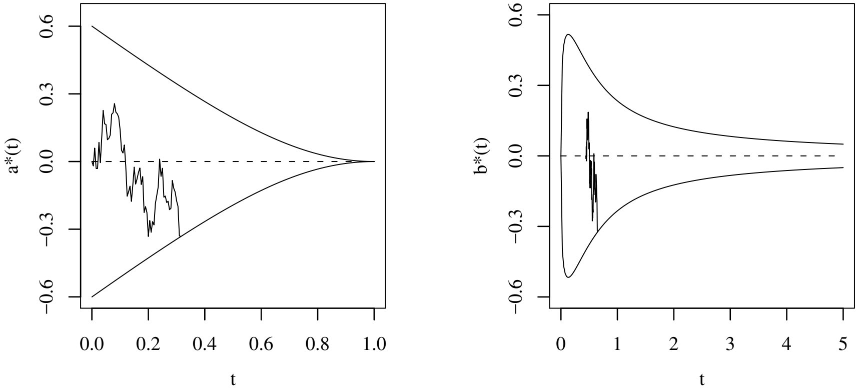
in the class of continuous functions $a(t)$ satisfying the property

$$0 < a(t) \leqslant \frac{\sigma_0^3}{4}(1-t) \text{ for } t < 1.$$

$$H(t,x) = \frac{1}{\sqrt{t}}\ \varphi(x/\sqrt{t}) - \frac{|x|}{t}\,\Phi(-|x|/\sqrt{t})).$$

The optimal stopping boundaries

Left: the boundary $a_1^*(t)$ for the process W.
Right: the boundary $b^*(t)$ for the process X.



Remarks

1. τ'^* is the solution of the problem

$$V = \inf_{\tau' \leqslant 1} \mathrm{E}\left[\frac{c}{1-\tau'} - |W_{\tau'}|\right], \qquad c = 2/\sigma_0^3,$$

2. As it follows from the construction of W,

$$b^*(t) = \sigma_0 t \cdot a^*_{\sigma_0}\left(1 - \frac{1}{\sigma_0^2 t}\right), \qquad t \geqslant \frac{1}{\sigma_0^2}.$$

Outline of the proof

For the average penalty we have

$$\begin{aligned}\mathrm{E}\big[\tau + |\mu|\mathbf{I}(d \neq \operatorname{sgn}(\mu))\big] \\ &= \mathrm{E}\big[\tau + E(\mu^- \mid \mathscr{F}_\tau)\mathbf{I}\{d = +1\} + \mathrm{E}(\mu^+ \mid \mathscr{F}_\tau)\mathbf{I}\{d = -1\}\big].\end{aligned}$$

Thus the problem is equivalent to

$$\mathscr{E}(\tau) := \mathrm{E}\big[\tau + \min\big\{\mathrm{E}(\mu^- \mid \mathscr{F}_\tau),\ \mathrm{E}(\mu^+ \mid \mathscr{F}_\tau)\big\}\big] \underset{\tau}{\to} \min\,.$$

Compute the conditional expectations:

$$\mathscr{E}(\tau) = \mathrm{E}[\tau + H(\tau + 1/\sigma_0^2,\ X_\tau)]$$

with the function

$$H(t,x) = \frac{1}{\sqrt{t}}\ \varphi(x/\sqrt{t}) - \frac{|x|}{t}\,\Phi(-|x|/\sqrt{t})).$$

Applying the Itô formula, we obtain

$$\mathscr{E}(\tau) = \frac{\sigma_0}{2} \mathrm{E}\left[\tau - \frac{|X_\tau|}{\sigma_0(\tau + 1/\sigma_0^2)}\right] + H\left(1/\sigma_0^2,\ 0\right).$$

Then we check that

$$M_t = \frac{X_t}{\sigma_0(t + 1/\sigma_0^2)} \quad \text{is a martingale.}$$

Applying the change of time, we find that

$$W_t = M_{\frac{t}{\sigma_0^2(1-t)}} \quad \text{is a Brownian motion,}$$

which reduces the Chernoff problem to the optimal stopping problem

$$V = \inf_{\tau' \leqslant 1} \mathrm{E}\left[\frac{2}{\sigma_0^3(1-\tau')} - |W_{\tau'}|\right].$$

Auxiliary results: unbounded optimal stopping problems

Consider a Markovian optimal stopping problem

$$V(t,x) = \inf_{\tau \leqslant T-t} \mathrm{E} G(t+\tau, W_\tau + x).$$

Suppose that

G is bounded from above

$\mathrm{E} \sup_{s \geqslant 0} |G(t+s, W_s + x)| < \infty$ for any $x \in \mathbb{R}$, $t \geqslant 0$.

Then the solution of the optimal stopping problem exists and

$$\tau^* = \inf\{s \geqslant 0 : (t+s, W_s + x) \in D\}$$

where the stopping set

$$D = \{(t,x) : V(t,x) = G(t,x)\}.$$

Now consider the case when G is unbounded.

Theorem. Suppose that G is continuous for $t < T$ and there exists a set $D^0 \subset D$ such that

$$\sup_{\substack{x,t \\ \tau \leqslant \tau_{D_0}}} \big(\mathrm{E} G(t + \ t, W_\tau + x) - G(t, x) \big) < +\infty,$$

$$\mathrm{E} \sup_{s \leqslant \tau_{D_0}} \big| G(t + s, W_s + x) \big| < \infty \text{ for any } x \in \mathbb{R},\ t \geqslant 0.$$

Then the function $V(t, x) - G(t, x)$ is finite and

$$\tau^* = \inf\{s \geqslant 0 : V(t + s, W_s + x) - G(t + s, W_s + x) = 0\}.$$

In the Chernoff problem

$$V(t,x) = \inf_{\tau \leqslant 1-t} \mathrm{E} G(t+\tau, W_\tau + x), \qquad G(t,x) = \frac{c}{1-t} - |x|.$$

We check that the conditions of the theorem hold for

$$D_0 = \left\{ (t,x) : |x| \geqslant \frac{\sigma_0^2}{4}(1-t) \right\}$$

The function

$$V(t,x)-G(t,x)=\inf_{\tau\leqslant 1-t}\mathrm{E}\left[\frac{c}{1-(t+\tau)}-|W_\tau+x|\right]-\frac{c}{1-t}+|x|$$

is non-decreasing for $t\geqslant 0$ and $x\geqslant 0$. Therefore,

$$D(x)=\{(t,x):|x|\geqslant a(t)\}$$

where $a(t)$ is non-increasing some function on $[0,1]$ such that

$$a(t)\leqslant\frac{\sigma_0^2}{4}(1-t).$$

It can be proved that a is continuous .

Let $F(t,x) = V(t,x) - G(t,x)$. Applying the Ito formula with local time on curves,

$$\begin{aligned}
&\mathrm{E}F(1, W_{1-t} + x) = F(t,x) \\
&\quad + \mathrm{E}\int_0^{1-t} (F'_t + F''_{xx})(t+s, W_s + x)\mathbf{I}\{W_s + x \neq \pm a(t+s)\} ds \\
&\quad + \mathrm{E}\int_0^{1-t} F'_x(t+s, W_s + x)\mathbf{I}\{W_s + x \neq \pm a(t+s)\} dW_s \\
&\quad + \frac{1}{2}\mathrm{E}\int_0^{1-t} \Delta F'_x(t+s, a(t+s))\mathbf{I}\{W_s + x = a(t+s)\} dL_s^{a(\cdot)} \\
&\quad + \frac{1}{2}\mathrm{E}\int_0^{1-t} \Delta F'_x(t+s, -a(t+s))\mathbf{I}\{W_s + x = -a(t+s)\} dL_s^{-a(\cdot)},
\end{aligned}$$

Then we take $x = a(t)$.

The Kiefer–Weiss problem

We observe the process

$$X_t = \mu t + B_t,$$

where μ is an unknown real parameter.

For given $\varepsilon > 0$, $\alpha \in (0, 1/2)$ define the class $\Delta_{\alpha,\varepsilon}$ of decision rules (τ, d) such that

$$\mathrm{P}(d \neq \mathrm{sgn}(u) \mid \mu = u) \leqslant \alpha \text{ for any } |u| > \varepsilon.$$

The Kiefer–Weiss problem:

$$\sup_{u \in \mathbb{R}} \mathrm{E}(\tau \mid \mu = u) \xrightarrow[(\tau,d) \in \Delta_{\alpha,\varepsilon}]{} \min$$

Known results

Lai showed that in the Kiefer-Weiss problem

$$\tau^* = \inf\{t \geqslant 0 : |X_t| \geqslant a^*(t)\}, \qquad d^* = \operatorname{sgn}(X_{\tau^*})$$

He obtained an estimate of the growth rate of $a^*(t)$ when $t \to \infty$.

Theorem

An optimal stopping time (τ^*, d^*) in the Kiefer–Weiss problem is

$$\tau^* = \inf\{t \geqslant 0 : |X_t| \geqslant a^*(t+t_0)\}, \qquad d^* = \operatorname{sgn} X_{\tau^*},$$

where $a^*(t) > 0$ is a non-increasing function on $\mathbb{R}$, being the unique solution of the integral equation

$$\exp(-\varepsilon a(t) - \varepsilon^2 t/2) = \int_0^\infty \big[\Phi_s(a(s+t)-a(t)) - \Phi_s(-a(s+t)-a(t))\big]\,ds$$

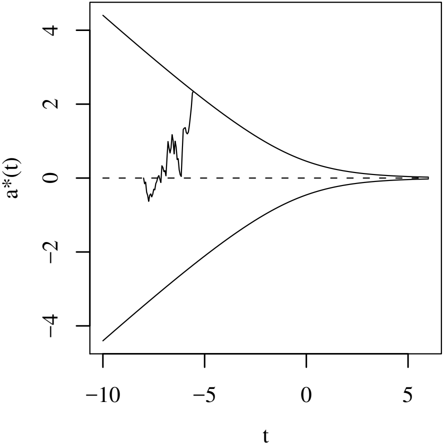
in the class of continuous function $a(t)$ on $\mathbb{R}$, satisfying the inequality

$$0 < a(t) \leqslant \varepsilon e^{-\varepsilon^2 t/2}/2, \qquad t \in \mathbb{R}.$$

The quantity $t_0 = t_0(\alpha)$ is found from the equation $\mathrm{P}(d^*(t_0) = 1 \mid \mu = -\varepsilon) = \alpha$.

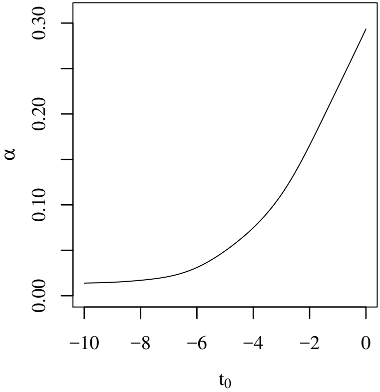
Numerical results

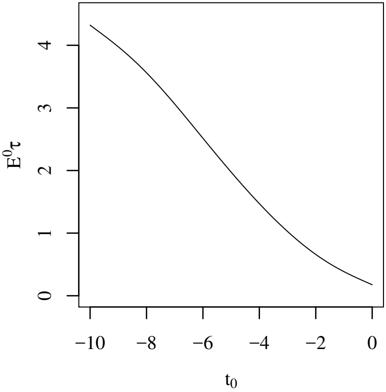
The optimal stopping boundary $a^*(t)$.



Left: the dependence of the probability α of a wrong decision on the value of t_0.

Right: the dependence of the maximal average observation time $\mathrm{E}^0\tau^*$ on t_0.





Outline of the proof

The problem is reduced to the family of problems V_c, $c > 0$:

$$V_c = \inf_{(\tau,d)} \big[\mathrm{E}(\tau \mid \mu = 0) + c\{\mathrm{P}(d = -1 \mid \mu = \varepsilon) + \mathrm{P}(d = 1 \mid \mu = -\varepsilon)\} \big].$$

If for $c > 0$ the optimal decision rule $\delta_c = (\tau_c, d_c)$ for V_c is such that

$$\mathrm{P}(d_c = -1 \mid \mu = \varepsilon) = \mathrm{P}(d_c = 1 \mid \mu = -\varepsilon) = \alpha,$$

then δ^c is optimal in the Kiefer–Weiss problem.

Using that $d(\mathrm{P}_t \mid \mu = u)/d(\mathrm{P}_t \mid \mu = u) = \exp(uX_t - u^2 t/2)$, we obtain

$$V_c = \inf_{(\tau,d)} \mathrm{E}\big[\tau + c\big(e^{\varepsilon X_\tau - \varepsilon^2\tau/2}\mathbf{I}\{d=-1\} + e^{-\varepsilon X_\tau - \varepsilon^2\tau/2}\mathbf{I}\{d=1\}\big) \mid \mu = 0\big].$$

This implies that the optimal decision rule (τ_c, d_c) is such that

$$d_c = \operatorname{sgn} X_{\tau_c}$$

and τ_c solves the optimal stopping problem

$$V_c = \inf_{\tau} \mathrm{E}\big[\tau + ce^{-\varepsilon|X_\tau| - \varepsilon^2\tau/2} \mid \mu = 0\big].$$

The constant $c = c(\alpha)$ is found from the condition

$$\mathrm{P}(X_{\tau_c} > 0 \mid \mu = -\varepsilon) = \alpha.$$

Fractional Brownian motion case

Suppose we observe the process

$$X_t = \mu t + B_t^H, \qquad t \geqslant 0,$$

where B^H is a fractional Brownian motion with Hurst index H, μ is an unknown drift coefficient.

We consider the problem of sequentially testing the hypotheses

$$H_+ : \mu \geqslant 0 \text{ and } H_- : \mu < 0.$$

Integral transform

It is known that process

$$M_t(B^H) = c_H \int_0^t s^{1/2-H}(t-s)^{1/2-H} dM_s^H,$$

with normalizing constant c_H is P-a.s. well defined for all values of $H \in (0,1)$ and turns out to be a martingale with respect to the natural filtration and has quadratic variation equal to

$$\langle M(B^H)\rangle_t = t^{2-2H},$$

where

$$c_H = \left(\frac{\Gamma(3-2H)}{2H\Gamma(3/2-H)^3\Gamma(1/2+H)} \right)^{1/2}.$$

Reduction of the problem

Process

$$M_t(X) = c_H \int_0^t s^{1/2-H}(t-s)^{1/2-H} dX_s$$

can be rewritten as

$$M_t(X) = M_t(B^H) + b_H \mu t^{2-2H}$$

or

$$M_t(X) = M_t(B^H) + b_H \mu \langle M(B^H) \rangle_t$$

with $b_H = c_H B(3/2 - H, 3/2 - H)$.

Equivalent problem

The process

$$B_t = M_{t^{1/(2-2H)}}(B^H),$$

is a standard Brownian motion and

$$X'_t = M_{t^{1/(2-2H)}}(X) = B_t + \mu' t$$

is a Brownian motion with drift $\mu' = b_H \mu$.

But the payment becomes nonlinear

$$\tau \to \tau' = \tau^{1/(2-2H)}.$$

Thank you