ETH zürich

Tractable SPDE Models for Order Books

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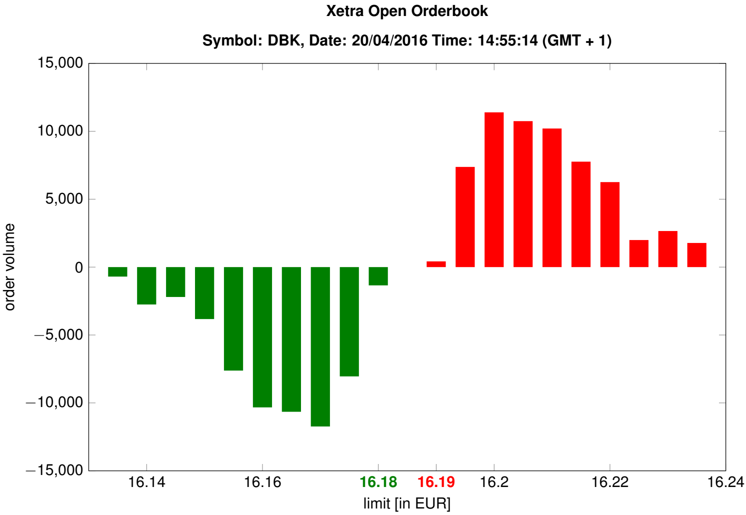
Limit Order Books



High-Dimensional Modeling

Mid price: $s := {}^{(p_{\text{bid}} + p_{\text{ask}})}/_{2}$

LOB Model: $v_t(p)$ density of LOB, centered: $u_t(p) := v_t(p + s_t)$

Observations and Assumptions

- HFT: $>$ 1000 orders per 10sec on average for some US stocks (Cont et al 2011)
- On average, orders arrive at prices p

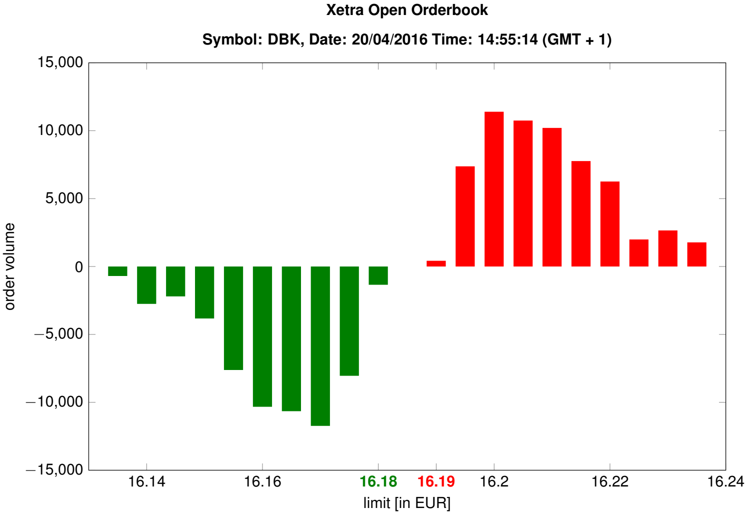
$$\sim a(b + |p - s_t|)^{-1-\mu}, \qquad \mu \in [0.6, 1.5]$$

(Bouchaud et al (2002), Zovko, Farmer (2006))

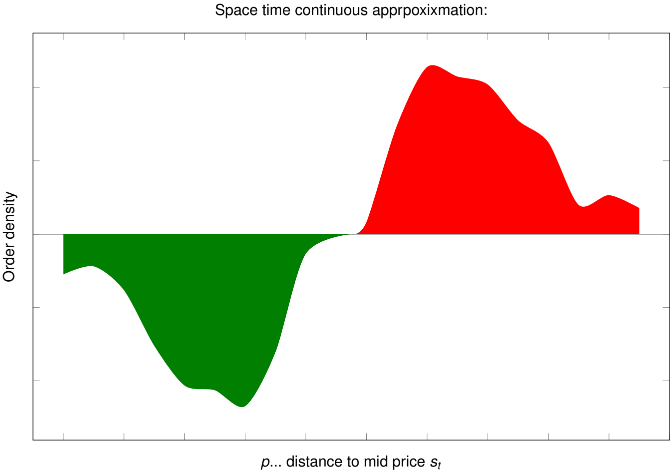
$\longrightarrow$ price-time-continuous model,

$\longrightarrow$ no spread, $p_{\text{bid}} = p_{\text{ask}} = s$.

Limit Order Books: Discrete Reality



Limit Order Books: Centered Density $u_t(p)$



Motivation

Since 2000, many authors studied (stochastic) PDEs for demand & supply / order book modeling, often in terms of approximation or MFG results.

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Since 2000, many authors studied (stochastic) PDEs for demand & supply / order book modeling, often in terms of approximation or MFG results.
Here:

1. Directly set up *macroscopic* description of order book dynamics in highly liquid markets.
2. Keep assumptions basic, focus on *tractability*.
3. *Calibration*!

Macroscopic Order Book Dynamics

1. Small order readjustments of HF-traders: rate η
2. Tendency to shift orders in direction to bid/ask (HF-traders): rate β
3. Net impact rate for order volume, of LF and HF-traders: α
4. LF-net impact due to exogeneous information: $g(p)$
5. HF-trader impact on volume: $\mathrm{d}M_t$

Order book density

With $\mathrm{d}p,\ \mathrm{d}t \to 0$, we impose for the centered order book density

$$\mathrm{d}u_t(p) = \underbrace{[\eta_a \Delta u_t(p) + \beta_a \nabla u_t(p) + \alpha_a u_t(p) + g_a(p)]}_{=A_a u_t + g_a} \mathrm{d}t + u_t(p)\ \mathrm{d}M_t^a.$$

for $p > 0$ (< 0 resp.) where M^a is a $\mathbb{R}$-valued cts local martingale.

Linear Models

On an abstract level,

$$\begin{cases} \mathrm{d}u_t(p) = A_a u_t(p)\,\mathrm{d}t + u_t(p)\,\mathrm{d}M_t^a, & p \in (0, L), \\ \mathrm{d}u_t(p) = A_b u_t(p)\,\mathrm{d}t + u_t(p)\,\mathrm{d}M_t^b, & p \in (-L, 0). \end{cases} \tag{1}$$

M^a, M^b are cts loc martingales and A_a, A_b are linear (unbounded) maps s. t. $\exists!$ strong solution, for reasonable initial data u_0.

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Theorem (Cont, Keller-Ressel, M. (2017))

The unique solution of (1) *is*

$$u_t(p) = g_t(p)\left[\mathcal{E}_t(M^b)\mathbf{1}_{(-L,0)}(p) + \mathcal{E}_t(M^a)\mathbf{1}_{(0,L)}(p)\right].$$

where g solves (1) *for $M^{a/b} \equiv 0$.*
In particular, u is a local martingale iff $Ah = 0$.

Spectral Analysis

Imposing Dirichlet bdry cond., the real eigenvalues of $A := \eta\Delta + \beta\nabla + \alpha$ on $(0, L)$ are

$$\nu_k := \alpha - k^2 \frac{\eta\pi^2}{L^2} - \frac{\beta^2}{4\eta}, \qquad k \in \mathbb{N},$$

for resp. eigenfcts

$$e_k(p) := e^{-\frac{\beta}{2\eta}p} \sin\left(\frac{k\pi}{L}p\right).$$

Observation

The only positive eigenfct is e_1 for the principle eigenvalue $\nu := \nu_1$.

$$
\begin{cases}
\mathrm{d}u_t(p) = \left[\eta^a \Delta u_t(p) + \beta^a \nabla u_t(p) + \alpha^a u_t(p)\right] \mathrm{d}t \\
\qquad\qquad\qquad\qquad + \sigma^a u_t(p)\, \mathrm{d}W_t^a, \quad p \in (0, L), \\
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\qquad\qquad\qquad\qquad + \sigma^b u_t(p)\, \mathrm{d}W_t^b, \quad p \in (-L, 0), \\
u_t(0+) = u_t(0-) = u(-L) = u(L) = 0, \\
\ u_t(p) > 0, \quad p \in (0, L), \ u_t(p) < 0, \quad p \in (-L, 0), \quad t > 0,
\end{cases}
\tag{1}
$$

From parametrization theorem: $u_t(p)$ explicitly computable

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From parametrization theorem: $u_t(p)$ explicitly computable, and

Corollary

For $u_0(p) = h(p) := \sin(\frac{\pi}{L} p) \exp\left(\pm \frac{\beta^{b/a}}{2\eta_{b/a}} p\right)$ the unique solution is

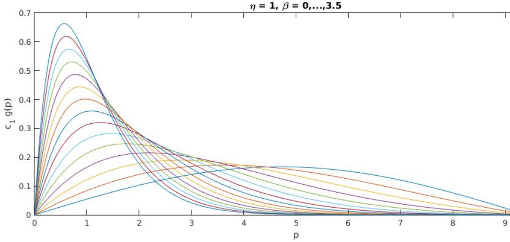
$$
u_t(p) = h(p) \left(X_t^b \mathbf{1}_{(-L,0)}(p) + X_t^a \mathbf{1}_{(0,L)}(p)\right),
$$

where $\mathrm{d}X_t^{a/b} = \nu_{a/b} X_t^{a/b}\, \mathrm{d}t + \sigma_{a/b} X_t^{a/b}\, \mathrm{d}W_t^{a/b}$

Order Book Shape

$$u_0(p) := \begin{cases} e^{\frac{\beta^b}{2\eta^b} p} \sin(\frac{\pi}{L} p), & p < 0 \\ e^{-\frac{\beta^a}{2\eta^a} p} \sin(\frac{\pi}{L} p), & p > 0. \end{cases}$$

$$\Rightarrow \quad u_t(p) = u_0(p) e^{\sigma_{a/b} W_t^{a/b} + \left(\nu_{a/b} - \frac{1}{2}\sigma_{a/b}^2\right) t},$$



Calibration

- Profile attains its maximum at:

$$\hat{p}_a := \operatorname{argmax}_p g^a(p) = \frac{L}{\pi} \arctan\left(\frac{2\eta_a \pi}{\beta_a L}\right) \approx \frac{2\eta_a}{\beta_a}$$

$\Rightarrow \frac{\beta}{2\eta}$ is determined by initial data!

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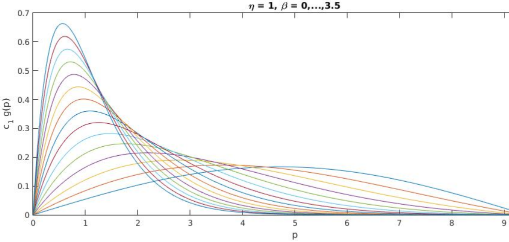
$$\Rightarrow \quad u_t(p) = u_0(p)e^{\sigma_{a/b}W_t^{a/b}+\left(\nu_{a/b}-\frac{1}{2}\sigma^2_{a/b}\right)t},$$



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$\Rightarrow$ $\frac{\beta}{2\eta}$ is determined by initial data!

- $\int_0^L u_t(p)\,\mathrm{d}p = X_t^a \frac{4\pi L\eta^2}{L^2\beta^2+4\eta^2\pi^2}\left(e^{-\frac{\beta}{2\eta}L}+1\right).$

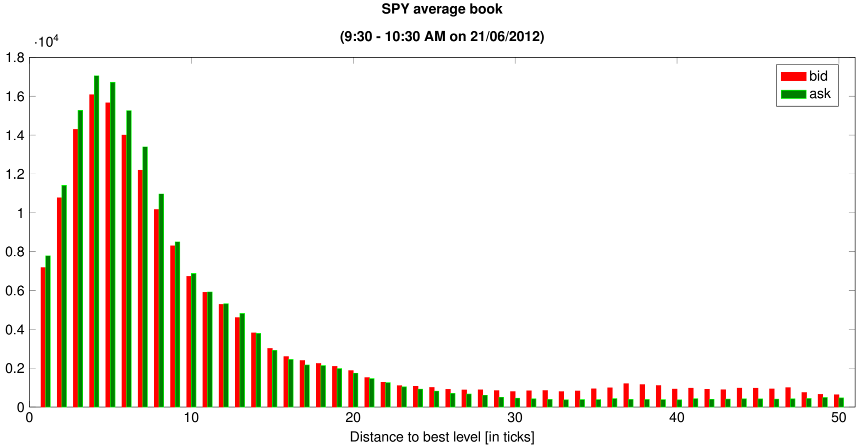


Figure: SPDR S&P 500 ETF NASDAQ data provided by LOBster

Total Order Volume in the Book

Volume in the book at time t is

$$\mathrm{TV}_t := \int_{-L}^{L} |u_t(p)| \, \mathrm{d}p = \mathrm{const}(\eta_a, \beta_a, L) X_t^a + \mathrm{const}(\eta_b, \beta_b, L) X_t^b,$$

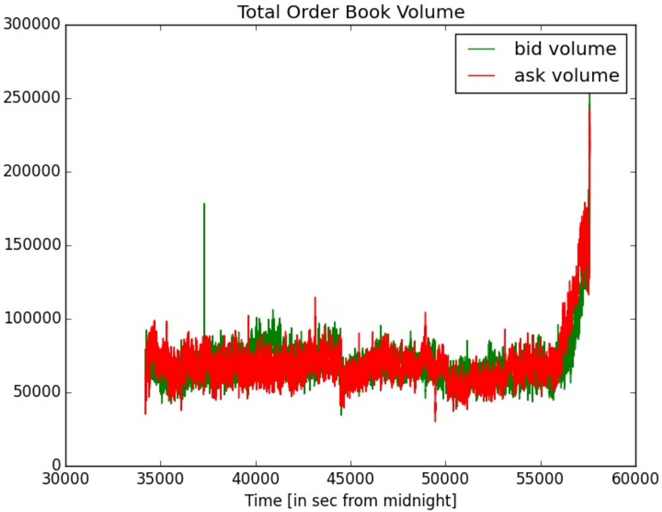
where $X_t = \exp\left(\sigma W_t + \left(\nu - \frac{1}{2}\sigma^2\right) t\right)$.

Observation:

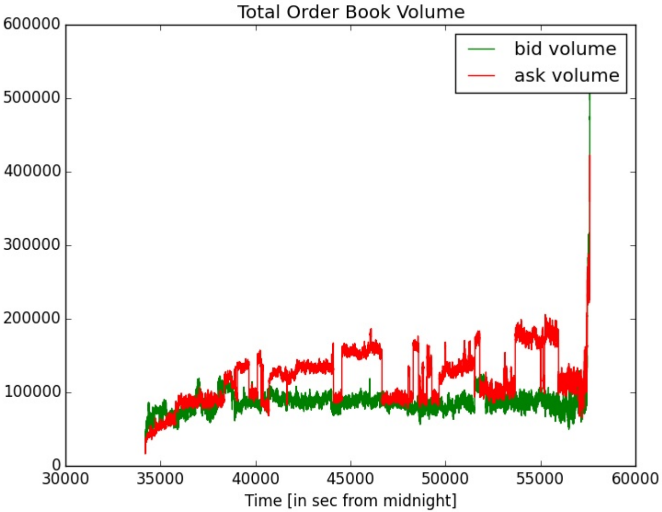
u_t and TV_t are martingales, iff

$$\boldsymbol{\alpha_{a/b} = \frac{\beta^2}{4\eta} + \frac{\pi^2 \eta}{L^2}}$$

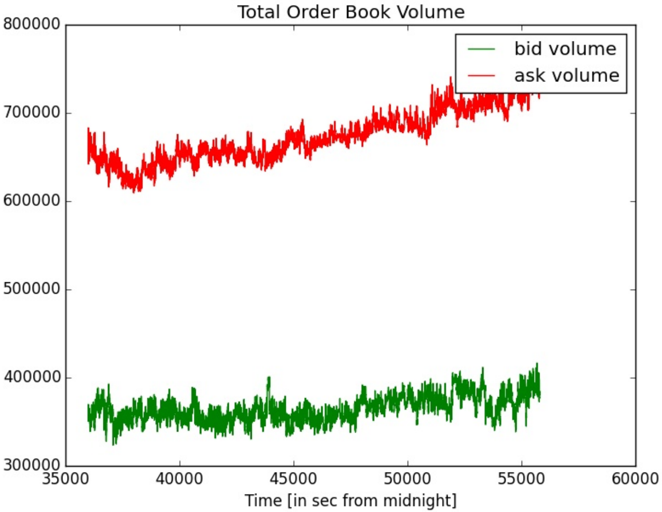
SPY 2016-06-13, first 50 levels:



MSFT 2016-09-13, first 50 levels:



`INTC` 2015-10-06, first 200 levels:



Linear Inhomogeneous Models

Now, keep $f(p) := \exp\left(-\frac{\beta}{2\eta}p\right)\sin\left(\frac{\pi}{L}p\right)$ and consider the inhom. model

$$\mathrm{d}u_t(p) = \left[\eta_{a/b}\Delta u_t(p) + \beta_{a/b}\nabla u_t(p) + \alpha_{a/b}u_t(p) + \lambda_{a/b}f_{a/b}(p)\right]\,\mathrm{d}t \\ + \sigma_{a/b}u_t(p)\,\mathrm{d}W_t^{a/b}$$

Compared with the hom. model we hope to get

- Mean reversion for “right” choice of parameters?
- Invariant distribution, ergodicity?
- Similiar parametrization?

Linear Inhomogeneous Models II

Consider now the one-sided problems,

$$\mathrm{d}u_t(p) = [Au_t(p) + \lambda f(p)]\,\mathrm{d}t + \sigma u_t(p)\,\mathrm{d}W_t, \qquad u_0(p) = z_0 f(p), \qquad (2)$$

$p \in (0, L)$, with initial data $z_0 \in \mathbb{R}$.

Theorem (Cont, Keller-Ressel, M. (2017))

- *If f is an eigenfct for A with eigenvalue $-\nu$, then*

$$u_t(p) = f(p)Z_t,$$

where $Z_0 = z_0$ and

$$\mathrm{d}Z_t = [\lambda - \nu Z_t]\,\mathrm{d}t + \sigma Z_t\,\mathrm{d}W_t.$$

- *If f is not an eigenfct, then no “reasonable” parametrization!*

The polynomial processes of type

$$dZ_t = \nu\,[\mu - Z_t]\,dt + \sigma Z_t\,dW_t.$$

are known in the frameworks of Pearson diffusions, GARCH-diffusion models (stoch vol), and if $\nu > 0$ in the notation:

$$dZ_t = \nu\left[\frac{\alpha}{\beta - 1} - Z_t\right]dt + \sqrt{\frac{2\nu}{\beta - 1}Z_t^2}\,dW_t,$$

as reciprocal gamma diffusion.

Prop

- Z_t is ergodic, stationary distribution is invers gamma with scale and shape parameters $\alpha > 0$ and $\beta > 1$, resp.
- $\mathbb{E}\,|Z_t|^k < \infty$ for all $k \leq k^{\max}$, iff $k^{\max} < \beta = 1 + \frac{2\nu}{\sigma^2}$.
- explicit solution for $Z_0 > 0$
- autocorellation $\mathbb{E}\,Z_{t+s}Z_t = e^{-\nu t}$.

Calibration

Estimators for the parameters have been studied by many authors, e. g. Leonenko and Suvak (2010):

Method of moments

Given observations $(\bar{Z}_t)_{t=1}^N$, set $\bar{m}_1 := \frac{1}{N}\sum_{t=1}^N \bar{Z}_t$, $\bar{m}_2 := \frac{1}{N}\sum_{k=1}^N \bar{Z}_t^2$.
Then,

$$\hat{\alpha} := \frac{\bar{m}_1 \bar{m}_2}{\bar{m}_2 - \bar{m}_1}, \qquad \hat{\beta} := 1 + \frac{\bar{m}_2}{\bar{m}_2 - (\bar{m}_1)^2}.$$

are P-consistent estimators, if $\beta > 4$.

Martingale estimating function for discr. observed diffusion processes (Bibby, Sørensen (1995), Leonenko and Suvak (2010)) yields P-cons. estimator for autocorrelation parameter ν (given α, β).

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$\Longrightarrow$ $\hat{\mu} := \frac{\hat{\alpha}}{\hat{\beta}-1}$, $\hat{\sigma} := \sqrt{\frac{\hat{\nu}}{\hat{\beta}-1}}$.

Price Prediction

- Empirical observation (Cont et al. '13):

$$\mathrm{d}s_t^{b/a} \approx \pm \frac{\mathrm{OF}_{b/a}(t)}{D_{b/a}(t)}, \qquad D \ldots \text{depths}, \quad \mathrm{OF} \ldots \text{order flow.}$$

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- First order approx for tick size δ,

$$D_{a/b}(t) = \pm \int_0^{\delta} u_t(\pm p)\, dp \approx \frac{\delta}{2} \nabla u_t(0\pm) = \frac{\delta \pi}{2L} Z_t^{a/b}, \tag{3}$$

$$\mathrm{OF}_{b/a}(t) \approx dD_{b/a}(t) \tag{4}$$

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$$\mathrm{OF}_{b/a}(t) \approx \mathrm{d}D_{b/a}(t) \tag{4}$$

Induced Price Model I

$$\mathrm{d}s_t = \frac{1}{2}\left(\mathrm{d}s_t^b + \mathrm{d}s_t^a\right) = \frac{1}{2}\left(\frac{\mathrm{d}Z_t^b}{Z_t^b} - \frac{\mathrm{d}Z_t^a}{Z_t^a}\right),$$

where Z^a and Z^b are the factor processes from before.

Price Dynamics

Induced Price Model II

Summarizing the price dynamics are

$$ds_t = \frac{1}{2}\left(\frac{\mu_b}{D_b(t)} - \frac{\mu_a}{D_a(t)} + (\nu_b - \nu_a)\right) \mathrm{d}t + \sigma_b \, \mathrm{d}W_t^b - \sigma_a \, \mathrm{d}W_t^a,$$

$$\mathrm{d}D_{a/b}(t) = \nu_{a/b}(\mu_{a/b} - D_{a/b}(t)) \, \mathrm{d}t + \sigma_{a/b} D_{a/b}(t) \, \mathrm{d}W_t^{a/b}$$

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$$\mathrm{d}D_{a/b}(t) = \nu_{a/b}(\mu_{a/b} - D_{a/b}(t))\,\mathrm{d}t + \sigma_{a/b}D_{a/b}(t)\,\mathrm{d}W_t^{a/b}$$

- Time inhom. extension of classical Bachelier model, based on empirical observations!
- $D_{b/a}^{-1} =: Y$ is unique solution of *logistic* SDE

$$\mathrm{d}Y_t = Y_t(\nu - \mu Y_t)\,\mathrm{d}t + \sigma Y_t\,\mathrm{d}W_t, \qquad Y_0 = \frac{1}{D(0)}$$

Calibration: Setup

Model for depths:

$$dD_{a/b}(t) = \nu_{a/b}(\mu_{a/b} - D_{a/b}(t))\, dt + \sigma_{a/b}\, dW_t^{a/b}, \tag{5}$$

with $[W^a, W^b]_t = \rho t$. First try:

1. Split trading day in 50ms time intervals
   $\rightarrow$ observations $D_a(t_i)$, $D_b(t_i)$ of depths in first two levels, $i = 1, ..., N$.
2. For t_i estimate parameters based on $D_a(t_j)$ and $D_b(t_j)$,
   $t_j \in [t_i - 30\text{min}, t_i]$.
3. Recalibrate at $t_{i+2} = t_i + 100ms$.

Run on NASDAQ data for some of most liquid large-tick stocks and SPY.

Summary

- General class of linear models
- Specific, simple subclass admits explicit solutions and fits average profiles very well (at least close to bid/ask)
  $\Longrightarrow$ Empirical justification on assumptions on the dynamics
- Linear inhomogeneous models reproduce mean-reversion of volume / depths, which is observed in the data
- Induced price models for prediction of price moves
- Fast calibration!