Tightness of a Family of Double Stochastic Pseudo-Poissonian Processes and Applications to a Real World Data

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Pseudo-Poisson Process as a subordinator

Let $(\xi) = \xi_0, \xi_1, \ldots$ be a random sequence;
$\Pi(s)$, $s \geq 0$, be the independent of (ξ) standard Poisson process with an intensity $\lambda > 0$.
We define a subordinator for a "forming" sequence (ξ) with a "leading" Poisson process by the following random change of the discrete mathematical time

$$\psi(s) = \psi_\Pi(s) = \xi_{\Pi(s)}, \qquad s \geq 0\,. \tag{1}$$

Poisson subordinator driving the discrete time of a Markov sequences is given and examined in the famous Feller's monograph (II Vol., Chap. X.). In this monograph such kind of subordinators are referred to the "Pseudo-poissonian processes".

Generally, for the arbitrary *forming* sequence (ξ) we name the process ψ as *the process of Poisson (random) Stochastic Index (process PSI)*

Spacings and their model interpretations

In fact, the process PSI marks the spacings of the leading Poisson process Π by terms of the forming sequence ξ.

Replacements: At every moment of a jump of the Poisson process the corresponding term (element) from the forming sequence is replaced with the next term.

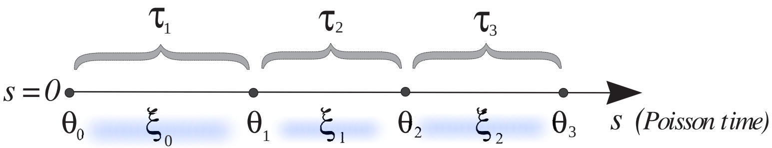


Figure: **Marked spacings**

The lengths of spacings $\tau_1, \tau_2, \ldots$ have to be i.i.d. random variables taking $\mathcal{E}xp(\lambda)$ distributed values.

Simple examples

- If $(\xi) = 0, 1, 2, \ldots,$ then the process PSI is equal in distribution to the Poisson process
- If $\xi_n = \sum_{j=1}^{n} \epsilon_j,$ (ϵ_j) is the sequence of i.i.d. random variables, $\sum_{j=1}^{0} = 0,$ then the Poisson psi-process is equal in distribution to the Compound Poisson process. In this well-known and examined case the process ψ has independent increments.
- If ξ consists of i.i.d. random variables with the symmetric binary distribution $\xi_0 = \pm 1$ with probability $1/2,$ then the process PSI is equal in distribution to the Telegraph process with the intensity $\lambda/2.$

Representation in the form of random weighted sums. Main covariance property

- The following representation in the form of random weighted sums with weights of the indicator type takes place

$$\psi_{\Pi}(s) = \sum_{j=0}^{\infty} \xi_j \mathbb{I}\{\Pi(s) = j\} \,. \qquad (2)$$

- Suppose, that $\mathbb{E}\xi_0 = 0$, $\mathbb{D}\xi_0 = 1$, $\xi_0, \xi_1, \ldots$ — i.i.d. rv's, then

$$\operatorname{cov}(\psi(s), \psi(u)) = \exp\{-\lambda(u - s)\} \qquad u \geq s \geq 0 \,. \qquad (3)$$

- **Remark** The exponential form (3) of the covariance is the same as for the Ornstein-Uhlenbeck covariance.

Covariance property in the arbitrary stationary case

Let the covariance function of (ξ) exist and be equal to $r(n), \quad n \in \mathbb{Z}_+$. In this case for $u \geq s \geq 0$

$$\operatorname{cov}(\psi(s), \psi(u)) = \mathbb{E}\{r(\Pi(u-s))\}. \qquad (4)$$

Sums of independent processes PSI for Gaussian limits

Let $\psi_1, \psi_2, \dots$ be independent copies of the process PSI $\psi(s)$, $s \geq 0$, when the forming sequence (ξ) consists of i.i.d. rv's, $\mathbb{E}\xi_0 = 0$, $\mathbb{D}\xi_0 = 1$. Consider the sums of (ψ) normalized by $\sqrt{N}$

$$\Psi_N(s) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \psi_i(s). \qquad (5)$$

The Main Functional Limit Theorem

Theorem 1. *Consider the piecewise constant random broken lines which are constructed by the values of the process* $\Psi_N(s)$*. Let us define them as elements of the Skorokhod space* $\mathcal{D}_{[0,\Theta]}$, $[0,\Theta] \ni s$, $\Theta \leq \infty$.

Then the following weak convergence in the Skorokhod space $\mathcal{D}_{[0,\Theta]}$ *takes place as* $N \to \infty$,

$$\Psi_N(s) \Longrightarrow U(s)\,, \tag{6}$$

where U *is the standard Ornstein-Uhlenbeck process: stationary, gaussian, markovian process with zero mean and with the variance which equals 1. Moreover,* $\mathrm{cov}(U(0), U(s)) = \exp\{-\lambda s\}$.

Wiener-Ornstein-Uhlenbeck (WOU) random field

Let us define the multi-index process, $t \in [0,1]$, $s \geq 0$, — prelimit random field:

$$\Psi_N(t,s) = \frac{1}{\sqrt{N}} \sum_{i=1}^{[Nt]} \psi_i(s)\,, \tag{7}$$

where $(\psi_i(s))$ are defined as for Theorem 1, $[\cdot]$ denotes the integer part. Let us introduce the Wiener-Ornstein-Uhlenbeck (WOU) random field $Z(t,s)$:

1) centered gaussian function defined on $\mathbb{R}_+ \times \mathbb{R}$;

2) $\mathrm{cov}(Z(t_1,s_1), Z(t_2,s_2)) = \exp\{-\lambda|s_2 - s_1|\} \min(t_1,t_2), \quad \lambda > 0.$

The WOU field is the tensor product of the Brownian motion and the Ornstein-Uhlenbeck process.

Convergence to the Wiener-Ornstein-Uhlenbeck Field

Theorem 2. *The following convergence of the finite dimensional distributions takes place as $N \to \infty$,*

$$\Psi_N(t,s) \Rightarrow Z(t,s) \qquad t \in [0,1],\ s \geq 0\,. \tag{8}$$

Moreover, the coefficient λ for Z is the same as the intensity coefficient of the leading Poisson process Π.

- The time t we name *the extrinsic time*.
- The time s we name *the intrinsic time*.

Fixing the value t^* of the extrinsic time we obtain the Ornstein-Uhlenbeck process with viscosity λ and with the variance of the truncations t.

Fixing the value s^* of the intrinsic we obtain the standard Brownian motion $Z(t,s^*) = W^{s^*}(t)$.

The Figure: the Bm of Residual Dependence, and Bm of Innovations

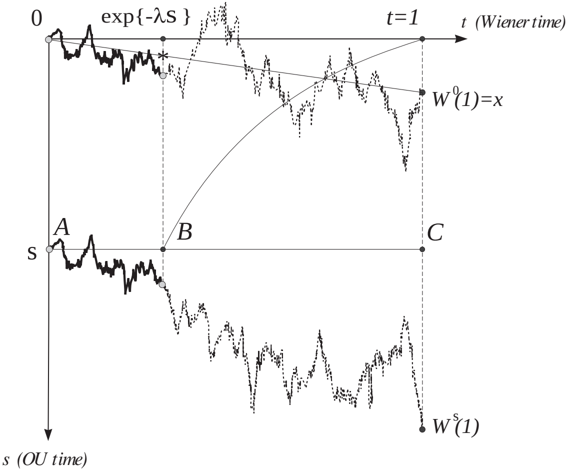


Figure: **Transition Characteristics and the Brownian Bridges**

Random Intensity for the Intrinsic Time

Definition. *Poison subordinator with a random intensity $\lambda(\omega)$ for the sequence (ξ), or Double Stochastic Pseudo-Poisson Process with a random intensity $\lambda(\omega)$ drawn at the initial time we define as follows*

$$\psi_{\lambda(\omega)} = \psi_{\lambda(\omega)}(s) = \psi(s;\, \lambda(\omega)) \stackrel{\triangle}{=} \xi_{\Pi_{\lambda(\omega)}(s)}, \qquad s \in \mathbb{R}_+\,, \tag{9}$$

where

$$\Pi_{\lambda(\omega)}(s) \stackrel{\triangle}{=} \Pi_1(s\lambda(\omega)), \quad s \in \mathbb{R}_+\,, \tag{10}$$

Π_1 *is the standardized Poisson process (with the intensity equals 1); the positive random variable $\lambda(\omega)$, (ξ), and Π_1 are jointly independent; a distribution function for $\lambda(\omega)$ we denote $F_{\lambda(\omega)}(x)$, $x \in (0, \infty)$.*

Laplace transform as a covariance

Lemma 1. *Suppose that the formed sequence* (ξ) *is i.i.d. Then the subordinator* $\psi_{\lambda(\omega)}(t)$ *is a strictly stationary process. Suppose that* $\mathbb{D}\xi_0 = 1$*, then [auto]covariance function is equal a the Laplace transform* $L_{\lambda(\omega)}(s),\ \ s \in \mathbb{R}_+$*, for the distribution function of* $\lambda(\omega)$

$$\operatorname{cov}\left(\psi_{\lambda(\omega)}(v), \psi_{\lambda(\omega)}(s+v)\right) = \int_0^\infty \exp\{-ys\}\mathrm{d}F_{\lambda(\omega)}(y) \tag{11}$$

Random Intensity of the Intrinsic Time follows to a Lévy process with positive a.s. increments. Covariance.

Lemma 2. *Let $L_{\Lambda(t)}(s)$, $s \geq 0$, denote a Laplace Transform over $s \in \mathbb{R}_+$ for a truncation of the random process $\Lambda(t)$ at any arbitrary but fixed moment of time $t > 0$. Then for i.i.d. (ξ) with $\mathbb{D}\xi_0 = 1$,*

$$\operatorname{cov}(\psi_\Lambda(u), \psi_\Lambda(v)) = L_{\Lambda(u-v)}(1), \quad u > v \geq 0\,, \tag{12}$$

i.e. in this case of random intensity we calculate the corresponding Laplace Transform at the point 1.

Curious Example. Suppose that $\Lambda(t)$ ia a (standard) subordinator in the "classical sense" (for Brownian motion), i.e. it is the Lévy process with positive α-stable increments, $0 < \alpha < 1$. Then it follows from Lemma 2, that

$$\operatorname{cov}(\psi_\Lambda(u), \psi_\Lambda(v)) = L_{\Lambda(u-v)}(1) = e^{-(u-v)}, \quad u > v \geq 0\,, \tag{13}$$

because in this case the Laplace Transform of $L_{\Lambda(t)}(s) = e^{-t\,s^\alpha}$. The OU covariance again!

Stationarity for Double Stochastic Pseudo-Poisson Process.

Case of stationarity (ξ).

Lemma 3. *Let (ξ) be a stationary sequence with $\mathbb{D}\xi_0 = 1$ and $R(n)$ be the covariance function for (ξ), $n \in \mathbb{Z}_+$. Then for $u \geq s \geq 0$,*

$$\begin{aligned}\operatorname{cov}(\psi_{\lambda(\omega)}(s), \psi_{\lambda(\omega)}(u)) &= \int_0^\infty \mathbb{E}R(\Pi_x(u-s))\mathrm{d}F_{\lambda(\omega)}(x) \\ &= \int_0^\infty \sum_{j=0}^{\infty} R(j)\frac{((u-s)x)^j}{j!}e^{-(u-s)x}\mathrm{d}F_{\lambda(\omega)}(x)\end{aligned} \tag{14}$$

Covariance Formula for Poissonian Subordinators for AR(1) Stationary Sequences

Assume that the forming sequence (ξ) is a stationary one. Then the corresponding process $\psi(s)$ is a stationary one, $s \geq 0$.

Proposition 1. Let the sequence (ξ) be a stationary of type of the AR(1) form

$$\xi_{n+1} = e^{-\gamma}\xi_n + \epsilon_{n+1} ,$$

where $\gamma > 0$, $n = 0, 1, \ldots,$ (ϵ_n) is a sequence of i.i.d. rv's; the sequence (ξ) is a centralized one, $\mathbf{E}\xi_n = 0$, $\mathbf{D}\xi_n = 1$. Then for $u \geq s \geq 0$,

$$\operatorname{cov}(\psi(s), \psi(u)) = \mathbb{E} \exp\{-\lambda(\omega)(1 - e^{-\gamma})(u - s)\} . \qquad (15)$$

Thus, in this case the corresponding covariance is a Laplace transform of $\lambda(\omega)(1 - e^{-\gamma})$ at the point $u - s$.

Gamma case for random intensity drawn at initial time

Fact 1. Assume that $\mathbb{E}\{\lambda^2(\omega)\} < \infty$. Then for $s \geq 0$

$$\begin{aligned} \mathbb{E}\Pi_{\lambda(\omega)}(s) &= s\mathbb{E}\{\lambda(\omega)\}, \\ \mathbb{D}\Pi_{\lambda(\omega)}(s) &= s\left(\mathbb{E}\{\lambda(\omega)\} + \mathbb{D}\{\lambda(\omega)\}\right). \end{aligned}$$

Let us consider the case of Γ-distribution for $\lambda(\omega)$ with a scale parameter $\gamma > 0$ (variable) and the (fixed) shape parameter $\kappa > 0$ (here for the Exponential distribution, i.e. for $\kappa = 1$, the density is $(1/\gamma)\exp(-t/\gamma)$, $t \geq 0$). In this case $\mathbb{E}\{\lambda(\omega)\} = \kappa\gamma$, and $\mathbb{D}\{\lambda(\omega)\} = \kappa\gamma^2$. Applying the calculated above expressions for $\mathbb{E}\Pi_{\lambda(\omega)}(s)$ and $\mathbb{D}\Pi_{\lambda(\omega)}(s)$, $s \geq 0$, one can obtain that:

$$\begin{aligned} \mathbb{E}\Pi_{\lambda}(s) &= s\kappa\gamma, \\ \mathbb{D}\Pi_{\lambda}(s) &= s\kappa\gamma + s\kappa\gamma^2. \end{aligned}$$

Random Intensity is a Random Scale Parameter for the Poisson Time Driven by a Positive Lévy Process

Consider a case for a random intensity when λ is a process Lévy with positive increments. So, $\lambda = \lambda(s, \omega)$ with a time parameter $s \geq 0$.
In this case the Lévy process $\lambda(s, \omega)$ generates an independently scattered random measure on $[0, \infty)$ with stationary increments and with the Lebesgue structural measure. The process $\Pi_{\lambda(s,\omega)}(s)$ is a homogeneous and a Markov one.
Fact 2. Assume that $\mathbb{E}\{\lambda^2(1, \omega)\} < \infty$. Then for $s \geq 0$

$$\begin{aligned} \mathbb{E}\Pi_{\lambda(s,\omega)}(s) &= \mathbb{E}\{\lambda(s, \omega)\}, \\ \mathbb{D}\Pi_{\lambda(s,\omega)}(s) &= \mathbb{E}\{\lambda(s, \omega)\} + \mathbb{D}\{\lambda(s, \omega)\}. \end{aligned}$$

Assume that $\lambda(s, \omega)$, $s \geq 0$, is the Γ-process Lévy with the corresponding space-scale parameter $\gamma > 0$ and the time-scale parameter $\kappa > 0$, i.e. $\lambda(1, \omega)$ has the $\Gamma(\gamma, \kappa)$ distribution possessing the infinitely divisible property. In this case

$$\begin{aligned} \mathbb{E}\Pi_{\lambda(s,\omega)}(s) &= s\kappa\gamma, \\ \mathbb{D}\Pi_{\lambda(s,\omega)}(s) &= s\kappa\gamma + s\kappa\gamma^2. \end{aligned}$$

Martingale Properties for the Considered “Double Stochastic Poisson Processes”

Lemma 4. *Let the random intensity* $\lambda(\omega)$ *be drawn at the initial moment of time and have a finite mean value. Let us define a filtration* $\{\mathcal{F}_1(s)\}_{s\geq 0}$ *as follows: for any fixed* $v \geq 0$ $\mathcal{F}_1(v) = \sigma(\Pi_1(r\lambda(\omega)),\ 0 \leq r \leq v)\}$. *Then the process* $\Pi_{\lambda(\omega)}(s) - s\mathbb{E}\{\lambda(\omega)\}$ *is a martingale with respect to* $\{\mathcal{F}_1(s)\}_{s\geq 0}$.

Let the random intensity $\lambda(s, \omega)$ *be driven by a positive Lévy process and have a finite mathematical expectation for some positive* s. *Let us define a filtration* $\{\mathcal{F}_2(s)\}_{s\geq 0}$ *as follows: for any fixed* $v \geq 0$ $\mathcal{F}_2(v) = \sigma(\Pi_1(\lambda(r, \omega)),\ 0 \leq r \leq v)\}$. *Then the process* $\Pi_{\lambda(s,\omega)}(s) - \mathbb{E}\{\lambda(s, \omega)\}$ *is a martingale with respect to* $\{\mathcal{F}_2(s)\}_{s\geq 0}$.

Reminder: Double Stochastic Pseudo-Poisson Process

Consider a PSI-process with random intensity: for a random intensity $\lambda = \lambda(\omega)$, a standard Poisson process $\Pi_1(t)$ and a sequence of i.i.d. random variables $\xi_0, \xi_1, \dots$ with $\mathbb{E}\xi_0 = 0$, $\mathbb{E}\xi_0^2 = 1$, all independent, define

$$\psi(t) = \xi_{\Pi_1(\lambda t)}.$$

Consider its independent copies ψ_k, $k = 1, \dots, N$, and define

$$\Psi_N(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \psi_k(t).$$

Note that the Poisson processes and intensities are different in each copy, so one should write

$$\Psi_N(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \xi^{(k)}_{\Pi_1^{(k)}(\lambda_k t)}.$$

Functional limit theorem for Double Stochastic Pseudo-Poisson Process

Impose the following restrictions:

- either $\mathbb{E}\lambda < \infty$ and $\mathbb{E}\xi_0^4 < \infty$;
- or $\mathbb{P}(\lambda > x) \sim cx^{-\gamma}$ as $x \to \infty$ for some $\gamma \in (0,1)$, $c > 0$, and $\mathbf{E}\xi_0^{4h} < \infty$ for some integer $h > 1/(2\gamma)$.

Theorem 3. In these settings there is a convergence $\Psi_N(t) \Rightarrow U_\lambda(t)$ in the Skorokhod space $\mathcal{D}_{[0,1]}$, with U_λ a Gaussian centered stationary process with the covariance function

$$\mathbb{E}U_\lambda(t)U_\lambda(s) = L_\lambda(|t-s|),$$

where L_λ is the Laplace transform of λ:

$$L_\lambda(t) = \mathbb{E}\{\exp(-t\lambda)\}, \qquad t \geq 0.$$

Sufficient conditions for convergence in $\mathcal{D}_{[0,1]}$ from Billingsley's book

Convergence in $\mathcal{D}_{[0,1]}$ takes place if

$$\mathbb{E}\{(\Psi_N(t) - \Psi_N(s))^{2h}(\Psi_N(u) - \Psi_N(t))^{2h}\} \leq C(u-s)^{1+\varepsilon}$$

for some $h > 0$ and $\varepsilon, C > 0$ and all $0 \leq s \leq t \leq u \leq 1$.

For fixed s, t, u the differences $\Psi_N(t) - \Psi_N(s)$ and $\Psi_N(u) - \Psi_N(t)$ are sums of independent random variables, but are dependent.

Many of these random variables are zero. This brings an idea to partition the set of indices $\{1, \ldots, N\}$ into 4 classes. This partition is random and independent of $\xi_i^{(k)}$ but depends on the Poisson processes $\Pi^{(k)}$ and intensities λ_k.

Random partitions for sums of PSI-processes

For fixed $s \le t \le u$ consider sets $A_{i,j}$, $i,j \in \{0,1\}$:

- $k \in A_{0,0}$ if $\Pi_k(\lambda_k t)$ has no jumps in $[s,u]$;
- $k \in A_{0,1}$ no jumps in $[s,t)$, jumps in $[t,u]$;
- $k \in A_{1,0}$ jumps in $[s,t)$, no jumps in $[t,u]$;
- $k \in A_{1,1}$ jumps in $[s,t)$, jumps in $[t,u]$.

Their sizes follow multinomial distribution:

$$\mathbb{P}\left(|A_{0,0}| = n_{0,0}, |A_{0,1}| = n_{0,1}, |A_{1,0}| = n_{1,0}, |A_{1,1}| = n_{1,1}\right)$$
$$= \begin{cases} N! \prod_{i,j=0}^{1} \dfrac{(p_{i,j})^{n_{i,j}}}{n_{i,j}!}, & n_{0,0} + n_{1,0} + n_{0,1} + n_{1,1} = N, \\ 0, & \text{otherwise.} \end{cases}$$

Only summands with indices from $A_{1,1}$ are dependent.

Random partitions for sums of PSI-processes II

The parameters $p_{i,j}$ can be expressed in terms of the Laplace transform of the random intensity λ:

$$\begin{aligned} p_{0,0} &= \mathbb{E}\big\{\mathbb{P}\big(\Pi(\lambda u) = \Pi(\lambda s)|\lambda\big)\big\} \\ &= \mathbb{E}\big\{\mathbb{P}\big(\Pi(\lambda(u-s)) = 0|\lambda\big)\big\} \\ &= \mathbb{E}\big\{e^{-\lambda(u-s)}\big\} \\ &= L_\lambda(u-s). \end{aligned}$$

Similarly,

$$\begin{aligned} p_{0,1} &= L_\lambda(t-s) - L_\lambda(u-s), \\ p_{1,0} &= L_\lambda(u-t) - L_\lambda(u-s), \\ p_{1,1} &= 1 - L_\lambda(t-s) - L_\lambda(u-t) + L_\lambda(u-s). \end{aligned}$$

This explains tail conditions on λ which can be also restated in terms of asymptotics of L_λ at 0.

Why moments of ξ are needed when λ has a heavy tail?

Example: Suppose both λ and ξ have heavy tails: for some $\gamma, \beta > 0$

$$\mathbf{P}(\lambda > x) \sim x^{-\gamma}, \qquad \mathbf{P}(\xi > x) \sim x^{-\beta}, \qquad x \to \infty.$$

Then for some $a \in (0,1)$ and large N

$$\mathbf{P}\big(\max\{\lambda_1, \ldots, \lambda_N\} > N^{1/\gamma}\big) > a.$$

Suppose it is $\lambda_k > n_\gamma := [N^{1/\gamma}]$, then

$$\mathbf{P}\big(\psi_k \text{ has at least } n_\gamma \text{ jumps on } [0,1]\big) > a/2.$$

Let $\max\{\xi_0^{(k)}, \ldots, \xi_{n_\gamma}^{(k)}\} = \xi_\ell^{(k)}$, then

$$\mathbf{P}\big(\xi_\ell^{(k)} > N^{1/(\beta\gamma)}\big) > b \in (0,1)$$

and its neighbours $\xi_{\ell-1}^{(k)}$ and $\xi_{\ell+1}^{(k)}$ are much smaller. If $1/(\beta\gamma) > 1/2$, that is $\beta < 2/\gamma$, then the jump of $\psi^{(k)}$ is not killed by scaling $1/\sqrt{N}$ of Ψ_N.

So one needs $\beta > 2/\gamma$ to get a continuous limit; we require slightly more: $\beta > 4k_0$ with $k_0 = \min\{k \in \mathbb{N} : 4k > 2/\gamma\}$.

Admiralty district

We process in dynamics a number of real data from real estate market in Saint-Petersburg, Russia. We analyze statistically empirical distributions of the real estate prises as well for input and output flows (stream) in the book of prices, as for truncations in time of the full population of objects of the real estate market. Statistical inferences are as follows:

- the hypotheses of Log-Normality of the prices are verified by the Kolmogorov-Smirnov (KS) test uniformly over more than 5 years of observations. Moreover, the "round" Kolmogorov-Smirnov test verifies a hypotheses of joint normality for logarithms of the input and output streams;
- accumulated averages increase linearly as well for the input stream, as for the output stream;
- accumulated variances increase very close to a linear grows as well for the input stream, as for the output stream.

These statistical inferences allows us in future to use the model of Poisson and "Double stochastic" Poisson subordinators for sequences with the goal to predict log prises in real estate markets.

Admiralty District of Saint-Petersburg: Fitting Log Normal.

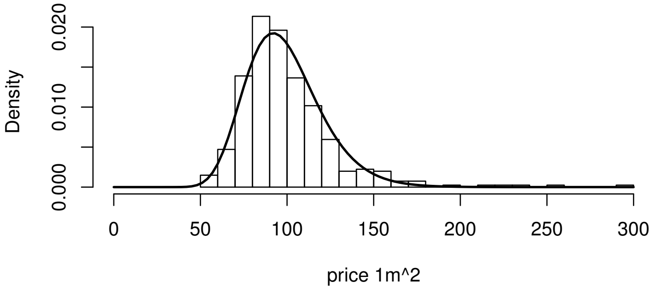
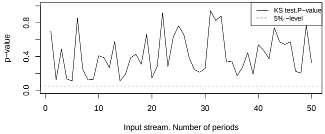
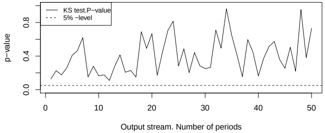
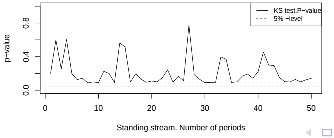


Figure: Admiralty district of Saint-Petersburg.

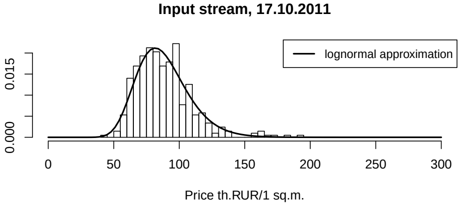
Admiralty District of Saint-Petersburg: p-values LN fit

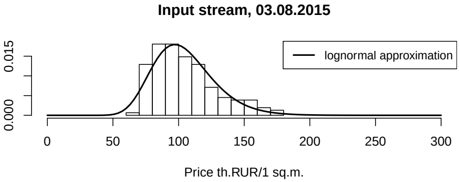




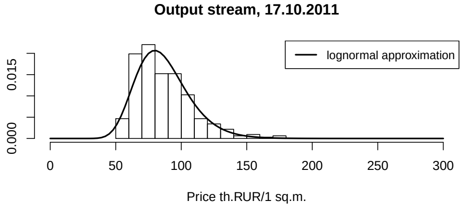


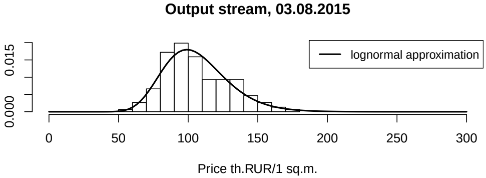
Initial and Final Input Flow Log Normal Approx



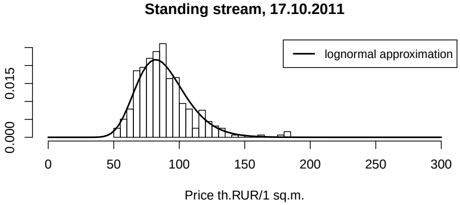


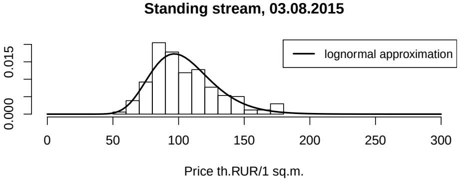
Initial and Final Output Flow Log Normal Approx



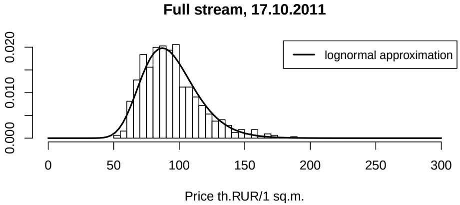


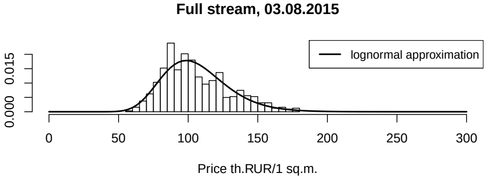
Initial and Final Standing Flow Log Normal Approx



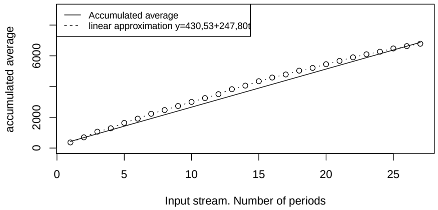


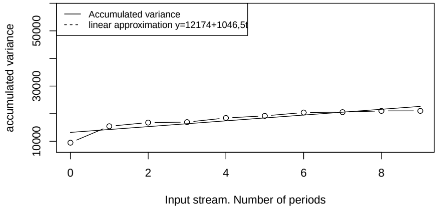
Initial and Final Full Flow Log Normal Approx



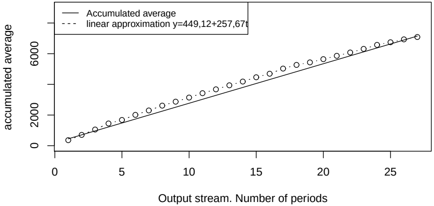


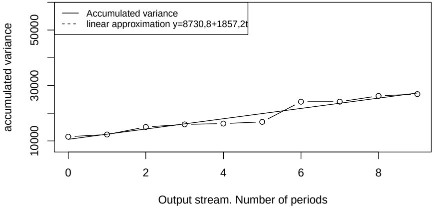
Cum Mean and Var for Input Flow



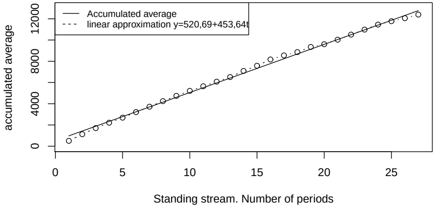


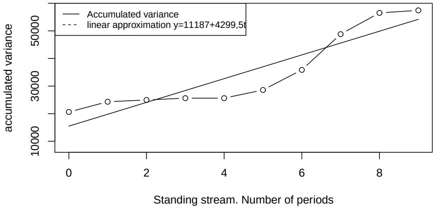
Cum Mean and Var for Output Flow





Cum Mean and Var for Standing Flow





Following to the Barndorff-Nielsen's representation we define a fractional Ornstein-Uhlenbeck process (fO-U) $U_H(t)$, $t \in \mathbb{R}$, of the Hurst parameter $H \in (0,1]$, as a Gaussian centered stationary process with the covariance

$$\begin{aligned} r(t) = r_H(t) &= \operatorname{cov}(U_H(0), U_H(t)) \\ &= \frac{1}{2}\left\{e^{-Ht} + e^{Ht} - |e^{t/2} - e^{-t/2}|^{2H}\right\}, \qquad t \in \mathbb{R}. \end{aligned}$$

Due to the Lamperti transform the process $w_H(s) \stackrel{\triangle}{=} s^H U_H(\log s)$, $s > 0$, $w_H(0) = 0$ a.s., is a fractional Brownian motion (fBm) process: Gaussian centered self-similar strictly stationary increments process with the Hurst parameter $H \in (0,1]$,

$$\operatorname{cov}(w_H(s), w_H(t)) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad s, t \geq 0.$$

For $t \geq 0$ the following chain of simple equalities allows us to obtain the expression for $r(t)$ in a form of the Laplace transform of a probability distribution,

$$r(t) = \frac{1}{2}e^{Ht}\left(1 + e^{-2Ht} - \left(1 - e^{-t}\right)^{2H}\right)$$

$$= \frac{1}{2}e^{-Ht} + \frac{1}{2}e^{Ht}\left(1 - \left(1 - e^{-t}\right)^{2H}\right)$$

$$= \frac{1}{2}e^{-Ht} + \frac{1}{2}\sum_{j=1}^{\infty}(-1)^{j-1}\binom{2H}{j}e^{-(j-H)t},$$

where

$$\binom{2H}{j} = \frac{\Gamma(2H+1)}{\Gamma(j+1)\Gamma(2H-j+1)}.$$

It is not difficult to check that in the case $0 < H < 1/2$ the last expression including t defines a fully monotone function equaling 1 at zero, hence it is the Laplace transform of a some probability distribution.

The following random variable ζ_H (for $0 < H < 1/2$) possesses this distribution of a discrete type

$$\mathbf{P}(\zeta_H = H) = \frac{1}{2} = p_0, \qquad \mathbf{P}(\zeta_H = j - H) = \frac{1}{2} p_j, \quad j \in \mathbb{N}\,;$$

$$p_1 = 2H\,, \qquad p_2 = \frac{2H(1-2H)}{2!}\,, \quad p_{k+1} = \left(1 - \frac{1+2H}{k+1}\right) p_k, \quad k \geq 2\,.$$

The random intensity $\lambda(\omega) \stackrel{d}{=} \xi_H$, substituted in definition of the DS PSI processes, provides that $\psi_H(s) = \xi_{\Pi_1(s\zeta_H)}$, $s \in \mathbb{R}_+$, has the same covariance as for fO-U.

Theorem 4. *Let us extend the stationary $\psi_H(t)$ on $\mathbb{R} \ni t$ and consider independent copies $\psi_H^{[j]}(t),\ \ j \in \mathbb{N},\ $ of $\psi_H(t)$ which subordinate (respectively) independent sequences $(\xi)^{[j]}$ of totally i.i.d random terms $\{\xi_i^{[j]}\},\ \ i \in \mathbb{Z}_+,\ \ j \in \mathbb{N}$, with $\mathbf{E}\xi_0^{[1]} = 0$, $\mathbb{D}\xi_0^{[1]} = 1$.*
Then the following convergence in a sense of convergence of finite dimensional distributions takes place as $N \to \infty$,

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \psi_H^{[j]}(t) \Rightarrow U_H(t)\,, \quad t \in \mathbb{R}\,;$$

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} s^H \psi_H^{[j]}(\log s) \Rightarrow W_H(s)\,, \quad s \in \mathbb{R}_+\,.$$

Proof of Theorem 4 directly follows from the Central Limit Theorem for vectors with the identical covariance.

Characteristics of the distribution of the random intensity which generates fBm

For analysis of the distribution ζ_H let us introduce the following random variable ζ, taking values on $\{0, 1, 2, \ldots\}$,

$$\begin{aligned}
\mathbf{P}(\zeta = 0) &= 2H\,, \\
\mathbf{P}(\zeta = 1) &= \frac{2H(1-2H)}{2!}\,, \\
\ldots &= \ldots\,, \\
\mathbf{P}(\zeta = k) &= \frac{2H(1-2H)(2-2H)\ldots(k-2H)}{(k+1)!}\,, \\
\ldots &= \ldots\,.
\end{aligned}$$

Obviously, the distribution of $\zeta + (1-H)$ equals (in Law) to a conditional distribution of ζ_H provided that $(\zeta_H \neq H)$.
After not difficult calculations we obtain a distribution function for ξ in the following explicit form

$$\mathbf{P}(\xi \le n) = 1 - \binom{n+1}{n+1-2H}\,, \qquad n \in \mathbb{N}\,,$$

and the asymptotic of its tail

$$\mathbf{P}(\xi \ge n) \sim \frac{n^{-2H}}{\Gamma(1-2H)}\,, \qquad n \to \infty\,.$$

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