Volatility and Arbitrage

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Motivation

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- What level of volatility is required and how long might it take for this arbitrage to be realized?
- A common but restrictive condition regarding market volatility, sometimes known as *strict nondegeneracy*, is the requirement that the eigenvalues of the market covariation matrix be bounded away from zero.
- This is a quite restrictive assumption regarding the behavior over time of the smallest eigenvalue of a random $d \times d$ matrix, where $d \in \mathbb{N}$ is usually a large integer (the number of stocks in an equity market).

Setup

- A probability space $(\Omega, \mathscr{F}, \mathrm{P})$ equipped with a right-continuous filtration $\mathfrak{F}$.
- $d \in \mathbb{N}$: number of assets at time zero. E.g., $d = 505$ (S&P 500) or $d = 4152$.

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- $d \in \mathbb{N}$: number of assets at time zero. E.g., $d = 505$ (S&P 500) or $d = 4152$.
- Nonnegative continuous semimartingales $\mu_1(\cdot), \cdots, \mu_d(\cdot)$, representing the relative market weight of each company.
- That is, $\mu_1(\cdot), \cdots, \mu_d(\cdot)$ are given by

$$\mu_i(t) = \frac{S_i(t)}{\sum_j S_j(t)},$$

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  where $S_j(\cdot)$ denotes the total capitalization of the j-th company.
- The process $\mu(\cdot)$ takes values in

$$\mathbf{\Delta}^d = \left\{ (x_1, \cdots, x_d)' \in [0,1]^d \ : \ \sum_{i=1}^{d} x_i = 1 \right\}.$$

An important empirical property of equity markets

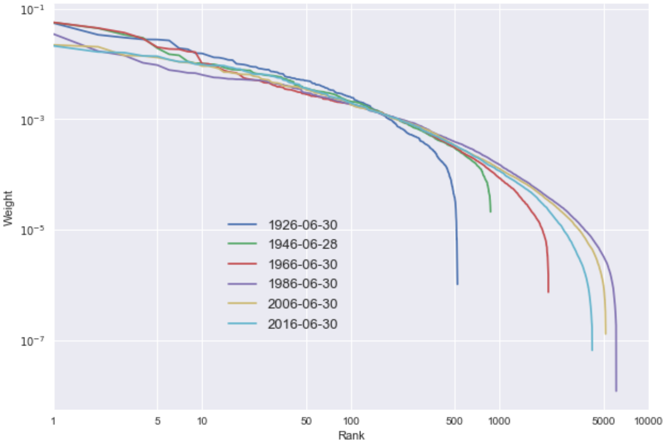

Figure: The capital distribution curve — Market weights $\mu_i(\cdot)$ against ranks on logarithmic scale, 1926–2016.

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$$G(\mu(t)) - G(\mu(0)) + \Gamma^G(t) = \int_0^t \langle \vartheta(s), \mathrm{d}\mu(s) \rangle,$$

where $\vartheta(s) = DG(\mu(s))$ and

$$\Gamma^G(t) = -\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t D^2_{ij} G\big(\mu(s)\big) \, \mathrm{d}\langle \mu_i, \mu_j \rangle(s).$$

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- The process $\Gamma^G(\cdot)$ is an aggregated cumulative measure of the market's internal variation.
- We shall formulate conditions on this process $\Gamma^G(\cdot)$ (instead of on the $(d-1)$–smallest eigenvalue of the covariation matrix).

Example: entropy function

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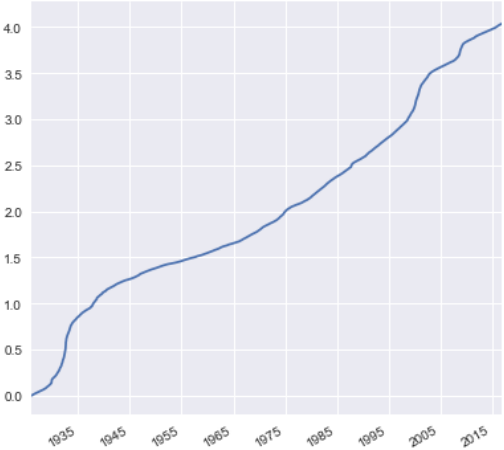
- Assuming $\mu(\cdot) \in \mathbf{\Delta}_+^d$ we have

$$\begin{aligned}\Gamma^H(\cdot) &= \frac{1}{2}\sum_{j=1}^{d}\int_0^{\cdot} \frac{\mathrm{d}\langle \mu_j, \mu_j\rangle(t)}{\mu_j(t)} \\ &= \frac{1}{2}\sum_{j=1}^{d}\int_0^{\cdot} \mu_j(t)\mathrm{d}\langle \log(\mu_j), \log(\mu_j)\rangle(t).\end{aligned}$$

The process $\Gamma^H(\cdot)$



Trading strategies

For an $\mathbb{R}^d$–valued predictable process $\vartheta(\cdot)$ write

$$V^{\vartheta}(t) := \sum_{i=1}^{d} \vartheta_i(t)\mu_i(t).$$

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Definition

Suppose that $\vartheta(\cdot)$ is integrable with respect to $\mu(\cdot)$ and that

$$V^{\vartheta}(T) - V^{\vartheta}(0) = \int_0^T \langle \vartheta(t), \mathrm{d}\mu(t) \rangle$$

holds. Then $\vartheta(\cdot)$ is called trading strategy.

Arbitrage relative to the market

Definition

A trading strategy $\vartheta(\cdot)$ is a relative arbitrage with respect to the market portfolio over the time horizon $[0, T]$ if

$$V^{\vartheta}(0) = 1; \qquad V^{\vartheta}(t) \geq 0$$

and

$$\mathrm{P}\Big(V^{\vartheta}(T) \geq 1\Big) = 1; \qquad \mathrm{P}\Big(V^{\vartheta}(T) > 1\Big) > 0.$$

Arbitrage over long term horizons

Let $G : \mathbf{\Delta}^d \to [0,\infty)$ be a concave C^2 function and recall the aggregated cumulative measure of the market's internal variation

$$\Gamma^G(t) = -\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^t D_{ij}^2 G\big(\mu(s)\big)\,\mathrm{d}\langle \mu_i, \mu_j\rangle(s).$$

Assume that for some constant $\eta > 0$

$$\mathrm{P}\Big(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^G(t) - \eta t \ \text{ is nondec.}\Big) = 1. \quad (*)$$

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Theorem

Assume (*). Then there exist $T^* > 0$ and a trading strategy $\varphi^G(\cdot)$ that is relative arbitrage with respect to the market over any time horizon $[0, T]$ with $T \in [T^*, \infty)$.

An important remark

As long as the market model $\mu(\cdot)$ satisfies, for some $\eta > 0$,

$$\mathsf{P}\left(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondec.}\right) = 1, \quad (*)$$

the arbitrage strategy

$$\varphi_i^G(\cdot) = D_i G\big(\mu(\cdot)\big) + \Gamma^G(\cdot) + G\big(\mu(\cdot)\big) - \sum_{j=1}^{d} \mu_j(\cdot) D_j G\big(\mu(\cdot)\big).$$

does not depend on the model parameters or the time horizon.

Existence of short-term arbitrage

If it's clear which of the components contributes to the overall variance:

Theorem

Suppose there exists a constant $\eta > 0$ *such that* $\langle \mu_1 \rangle(t) \geq \eta t$ *holds on the stochastic interval* $[\![0, \mathscr{D}^* [\![$ *with*

$$\mathscr{D}^* := \inf \left\{ t \geq 0 : \ \mu_1(t) \leq \frac{\mu_1(0)}{2} \right\}.$$

Then relative arbitrage exists over the time horizon $[0, T]$*, for every* $T > 0$.

Existence of short-term arbitrage (cont'd)

If we have full support:

Theorem

Suppose that for a given regular function G and appropriate real constants $\eta > 0$ and $h > 0$,

$$\mathrm{P}\left(\textit{the mapping } [0,\infty) \ni t \mapsto \Gamma^G(t) - \eta t \;\textit{ is nondec.}\right) = 1; \quad (*)$$

$$\mathrm{P}\big(G(\mu(t)) \geq h, \quad t \geq 0\big) = 1$$

and the "time homogeneous support" property

$$\mathrm{P}\Big(G\big(\mu(t)\big) \in \big[h, h+\varepsilon\big), \;\; \textit{for some } t \in [0,T]\Big) > 0, \quad \textit{for all } T, \varepsilon > 0.$$

Then relative arbitrage exists over the time horizon $[0, T]$, for every $T > 0$.

Existence of short-term arbitrage (cont'd)

Does the condition

$$\mathsf{P}\left(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondec.}\right) = 1 \qquad (*)$$

always imply the existence of short-term arbitrage?

Answer: No; there exist models that satisfy (*) but do not allow for short-term arbitrage.

A counter-example

- Consider the generating function

$$Q(x) = 1 - \sum_{j=1}^{d} x_j^2.$$

- Then

$$\Gamma^Q(\cdot) = \sum_{j=1}^{d} \langle \mu_j, \mu_j \rangle(\cdot)$$

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- Goal: Construct process $\mu(\cdot)$ with each component a martingale such that $\Gamma^Q(t) = t$, $t \in [0, T^*]$ for some $T^* > 0$.
- This then yields a counterexample.

An Itô diffusion

- Fix $d = 3$ (three assets).
- Consider SDEs:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}}(v_2(t) - v_3(t))\mathrm{d}\Theta(t);$$
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- Define $r(v) = \sqrt{\sum_{i=1}^{3} \left(v_i - \frac{1}{3}\right)^2}$.

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- Itô's formula yields

$$\langle v_1\rangle(t) + \langle v_2\rangle(t) + \langle v_3\rangle(t) = r^2(v(t)) = r^2(v(0))\mathrm{e}^t.$$

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- A solution:

$$v_i(t) = \frac{1}{3} + \delta\mathrm{e}^{t/2}\cos\left(\Theta(t) + 2\pi\left(u + \frac{i-1}{3}\right)\right).$$

An Itô diffusion (cont'd)

- A slight modification:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}r(t)}(v_2(t) - v_3(t))\mathrm{d}\Theta(t);$$
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- This can be made sense of also if

$$v_1(0) = v_2(0) = v_3(0) = 1/3.$$

- Now,

$$\langle v_1\rangle(t) + \langle v_2\rangle(t) + \langle v_3\rangle(t) = r^2(v(t)) = t.$$

- Market model $\mu(\cdot)$: stopped version of $v(\cdot)$.

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Conclusion

For a concave C^2 function G consider the condition

$$\mathsf{P}\left(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondec.}\right) = 1. \quad (*)$$

- Under (*), there exists T^* such that relative arbitrage over all $[0, T]$ with $T > T^*$ can be explicitly constructed.
- Under (*) and additional conditions (e.g., time-homogeneous support), for each T there exists relative arbitrage over $[0, T]$.
- Without additional conditions, (*) is not sufficient for the existence of short-term relative arbitrage.
- In particular, there exist $T > 0$ and $\boldsymbol{\Delta}^d$–valued $\mu(\cdot)$ such that $\mu(\cdot \wedge T)$ is a martingale but (*) holds.
- Such $\mu(\cdot)$ can be an Itô process with a covariation process whose $(d-1)$-st largest eigenvalue is strictly positive (but not bounded away from zero).

Some references

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Спасибо!