Optimal Consumption under Non-addictive Habit Formation in Incomplete Markets

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Outline

- Introduction and Literature Review
- Mathematical Model and Set-up
- Main Results
  - auxiliary processes and dual problems
  - path-dependent constraint and stochastic LM process
- Some Future Work

Consumption Habit Formation

- The standard Merton optimal consumption problem:

$$u(x) = \sup_{(H,c)\in\mathcal{A}_x} \mathbb{E}\left[\int_0^T U(t, c_t)dt\right],$$

where $\mathcal{A}_x$ is the admissible set of portfolio-consumption strategies (H, c) with the initial wealth $x > 0$.

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where $\mathcal{A}_x$ is the admissible set of portfolio-consumption strategies (H, c) with the initial wealth $x > 0$.

- However, some empirical studies argued that
  - the von Neumann-Morgenstern utilities can not reconcile the equity premium puzzles.
  - the consumer's satisfaction level and risk tolerance sometimes rely more on recent changes.
  - the smooth consumption is more beneficial than the marked increase, such as the household consumption and expenditures with commitment.

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  - the consumer's satisfaction level and risk tolerance sometimes rely more on recent changes.
  - the smooth consumption is more beneficial than the marked increase, such as the household consumption and expenditures with commitment.

- The utility function should not merely be defined on the consumption rate, but also on the history pattern of the whole consumption path.

Consumption Habit Formation

- The consumption habit formation preference is defined as

$$u(x) = \sup_{(H,c)\in\mathcal{A}_x} \mathbb{E}\left[\int_0^T U(t, c_t - Z(c)_t)dt\right],$$

where the accumulative process $Z(c)$ is called the habit formation process which satisfies the recursive equation

$$dZ(c)_t = (\delta_t c_t - \alpha_t Z(c)_t)dt, \qquad Z(c)_0 = z.$$

- Equivalently,

$$Z(c)_t = ze^{-\int_0^t \alpha_v dv} + \int_0^t \delta_s e^{-\int_s^t \alpha_v dv} c_s ds,$$

where discounting factors α_t and δ_t measure, respectively, the persistence of the initial habits level and the intensity of consumption history. In general, α and δ are assumed to be bounded optional processes.

Addictive Habits vs Non-addictive Habits

- Addictive Habit Formation: if $U : [0, T] \times (0, +\infty) \to \mathbb{R}$, i.e., it is required that $c_t \geq Z(c)_t$ at any $t \in [0, T]$.
  - Complete Market Model with Ito processes: Detemple and Zapatero (Econometrica 1991, MF 1992), Schroder and Skiadas (RFS 2002), Englezos and Karatzas (SICON 2009)
  - General Incomplet Market Models: Yu (AAP, 2015)
  - Market Models with Transaction Costs: Yu (AAP, 2017)

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- Non-addictive Habit Formation: if $U : [0, T] \times (-\infty, +\infty) \to \mathbb{R}$, the consumption rate is allowed to fall below the standard of living process.
  - Complete Market Model with Ito processes: Detemple and Karatzas (JET, 2003)
  - Incomplete Market Model: None. (Motivation of our research)

Market Model

- Let us consider d risky assets modelled by a d-dimensional locally bounded semimartingale $(S_t^{(1)}, \ldots, S_t^{(d)})_{t \in [0,T]}$ on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ and one riskless bond $S_t^{(0)} \equiv 1, \forall t \in [0, T]$ which is the numéraire asset.

- The self-financing wealth process $(W_t^{x,H,c})_{t \in [0,T]}$ is given by

$$W_t^{x,H,c} \triangleq x + (H \cdot S)_t - \int_0^t c_s ds, \quad t \in [0, T].$$

  The consumption policy c_t is called x-*financeable* if the no-bankruptcy condition is satisfied, i.e., $W_t^{x,H,c} \ge 0$ a.s. for $t \in [0, T]$. Let $\mathcal{A}_x$ denotes the set of x-financeable consumptions.

- $\mathcal{M}$ denotes the family of equivalent local martingale measures and $\mathcal{M} \neq \emptyset$.

Market Model

- The optional decomposition theorem implies the consumption budget constraint condition: the process $(c_t)_{t\in[0,T]}$ is x-financeable if and only if

$$\mathbb{E}\Big[\int_0^T c_t Y_t dt\Big] \leq x, \quad \forall Y_t \in \mathcal{M}.$$

- The **primal utility maximization problem** with non-addictive habit formation is defined as

$$u(x;z) \triangleq \sup_{c\in\mathcal{A}_x} \mathbb{U}(c) = \sup_{c\in\mathcal{A}_x} \mathbb{E}\Big[\int_0^T U(t, c_t - Z(c)_t)dt\Big], \quad x > 0, z > 0.$$

  where $U : [0, T] \times (-\infty, \infty) \to \mathbb{R}$ satisfies the standard conditions.

- Although the habit formation is not addictive, the non-negative consumption constraint $c_t \geq 0$ is active.

Duality Approach with Auxiliary Processes

- The path-dependent structure and potential time inconsistency may break the standard DPP argument.

- The feedback form is not expected in incomplete market models and special structures of the optimal consumption process are almost hopeless from the stochastic control approach.

- The classic duality between consumption rate process $c \in \mathcal{A}_x$ and the martingale measure density $Y \in \mathcal{M}$ does not work in our model due to the path integral term in $c_t - Z(c)_t$.

- We shall apply the duality approach using the auxiliary processes to hide the path-dependence.

Duality Approach with Auxiliary Processes

- **Step 1:** Treat $\tilde{c}_t = c_t - \int_0^t \delta_s e^{-\int_s^t \alpha_v dv} c_s ds$ as the auxiliary primal process and denote $\widetilde{\mathcal{A}}_x$ as the set of all $\tilde{c}$ for $c \in \mathcal{A}_x$.

- **Step 2:** Construct the auxiliary dual process

$$\Gamma_t \triangleq Y_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v) dv} Y_s ds \Big| \mathcal{F}_t \Big], \ \textit{for each } Y \in \mathcal{M}.$$

Denote $\widetilde{\mathcal{M}}$ the set of all Γ. We will have $\mathbb{E}\Big[\int_0^T c_t Y_t dt \Big] = \mathbb{E}\Big[\int_0^T \tilde{c}_t \Gamma_t dt \Big]$.

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- There are many challenges:
  - Duality in which space?
  - What about the extra term $ze^{-\int_0^t \alpha_v dv}$?
  - The nonnegative constraint on $c_t \geq 0$ mandates the path dependent constraint

$$\tilde{c}_t \geq -\int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \tilde{c}_s ds.$$

Duality Approach with Auxiliary Processes

- Trade off between c and $\tilde{c}$: No duality for c and path-dependent constraint on $\tilde{c}$.

- The auxiliary primal space can be written as

$$\widetilde{\mathcal{A}}_x = \Big\{ \tilde{c} \in \mathbb{L}^0 : \mathbb{E}\Big[\int_0^T \tilde{c}_t \Gamma_t dt \Big] \leq x, \quad \forall \Gamma \in \widetilde{\mathcal{M}}, \text{and with the constraint}$$
$$\tilde{c}_t \geq - \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \tilde{c}_s ds \Big\}.$$

- Consider the product space $[0, T] \times \Omega$ with the finite measure $d\widetilde{\mathbb{P}} = dt \times d\mathbb{P}$. Let $\mathcal{O}$ be the σ-algebra of optional sets relative to the filtration $(\mathcal{F}_t)_{t \in [0,T]}$

- The dual space $\widetilde{\mathcal{M}}$ is not closed in any sense.

Duality Approach with Auxiliary Processes

- Extend the space $\widetilde{\mathcal{M}}$ to the weak-$*$ $\sigma(\mathbf{ba}(\mathcal{O}, \widetilde{\mathbb{P}}), \mathbb{L}^\infty(\mathcal{O}, \widetilde{\mathbb{P}}))$ closure $\widetilde{\mathcal{D}}$, which is a set of bounded finitely additive measures $\widetilde{\mathbb{Q}}$ on $\mathcal{O}$.

- For each $x > 0$, we have an equivalent characterization of the set $\widetilde{\mathcal{A}}_x$,

$$\widetilde{\mathcal{A}}_x = \Big\{ \tilde{c} \in \mathcal{V}^{\widetilde{\mathcal{M}}} : \langle \tilde{c}, \widetilde{\mathbb{Q}} \rangle \leq x, \text{ for all } \widetilde{\mathbb{Q}} \in \widetilde{\mathcal{D}} \text{ and with the constraint}$$
$$\tilde{c}_t \geq -\int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \tilde{c}_s ds \Big\}.$$

- The **auxiliary primal utility maximization problem** is written as

$$\tilde{u}(x; z) \triangleq \sup_{\tilde{c} \in \widetilde{\mathcal{A}}_x} \mathbb{U}(\tilde{c}) = \sup_{\tilde{c} \in \widetilde{\mathcal{A}}_x} \mathbb{E}\Big[\int_0^T U(t, \tilde{c}_t - z \tilde{w}_t) dt \Big],$$

where we denote $\tilde{w}_t \triangleq e^{\int_0^t (-\alpha_v) dv}$ for all $t \in [0, T]$ as some shadow random endowments.

Duality Approach with Auxiliary Processes

- For each fixed Lagrange multipliers $y > 0$ and $\xi \in \mathbb{L}^0_+$, the **auxiliary dual optimization problem** is defined by

$$v(y,\xi) = \inf_{\widetilde{\mathbb{Q}} \in \widetilde{\mathcal{D}}(y)} \mathbb{V}(\widetilde{\mathbb{Q}}; y, \xi),$$

where we define the functional $\mathbb{V}(\widetilde{\mathbb{Q}}; y, \xi)$ as

$$\mathbb{V}(\widetilde{\mathbb{Q}}; y, \xi) \triangleq \sup_{\tilde{c} \in \widetilde{\mathcal{A}}_x} \Big(\mathbb{U}(\tilde{c}) - \big\langle \tilde{c} - z\tilde{w}, \widetilde{\mathbb{Q}} \big\rangle$$
$$+ \mathbb{E}\Big[\int_0^T \Big(\tilde{c}_t + \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \tilde{c}_s ds \Big) \xi_t dt \Big] \Big).$$

Duality Approach with Auxiliary Processes

- As a matter of fact, Fubini's theorem deduces that

$$\mathbb{E}\Big[\int_0^T \Big(\tilde{c}_t + \int_0^t \delta_s e^{\int_s^t(\delta_v-\alpha_v)dv}\tilde{c}_s ds\Big)\xi_t dt\Big] = \mathbb{E}\Big[\int_0^T \tilde{c}_t\tilde{\xi}_t dt\Big],$$

where $\tilde{\xi}_t \triangleq \xi_t + \delta_t\mathbb{E}\Big[\int_t^T e^{\int_t^s(\delta_v-\alpha_v)dv}\xi_s ds\Big|\mathcal{F}_t\Big]$ and satisfies

$$\tilde{\xi}_t \geq \delta_t\mathbb{E}\Big[\int_t^T \tilde{\xi}_s e^{\int_t^s(-\alpha_v)dv} ds\Big|\mathcal{F}_t\Big], \quad \text{a.s.} \quad \forall t \in [0, T].$$

- The dual functional can be written explicitly as

$$\mathbb{V}(\widetilde{\mathbb{Q}}; y, \tilde{\xi}) = \mathbb{E}\Big[\int_0^T V(t, -\tilde{\xi}_t + \Gamma_t^{\widetilde{\mathbb{Q}}})dt\Big] + \langle z\tilde{w}, \widetilde{\mathbb{Q}}\rangle + \mathbb{E}\Big[\int_0^T z\tilde{w}_t\tilde{\xi}_t dt\Big],$$

where $\Gamma^{\widetilde{\mathbb{Q}}}(t,\omega) = \frac{d\widetilde{\mathbb{Q}}^r}{d\widetilde{\mathbb{P}}}$ and $\widetilde{\mathbb{Q}} = \widetilde{\mathbb{Q}}^r + \widetilde{\mathbb{Q}}^s \in \widetilde{\mathcal{D}}(y)$.

Duality Approach with Auxiliary Processes

- To build the duality between $\tilde{u}(x)$ and $v(y, \tilde{\xi})$: How to choose the stochastic Lagrange multiplier $\tilde{\xi}^*$?

- The answer depends on another two auxiliary problems: the unconstrained auxiliary primal and dual problems.

Unconstrained Auxiliary Problems

- Consider the enlarged admissible space for the auxiliary primal space $\widetilde{\mathcal{A}}_x$ where we consider all $x \in \mathbb{R}$,

$$\bar{\mathcal{A}}_x = \left\{ \bar{c} : \ \langle \bar{c}, \widetilde{\mathbb{Q}} \rangle \leq x, \ \text{for all } \widetilde{\mathbb{Q}} \in \widetilde{\mathcal{D}} \right\}, \qquad x \in \mathbb{R}.$$

  The **auxiliary unconstrained primal utility maximization problem** is defined as

$$\bar{u}(x) = \sup_{\bar{c} \in \bar{\mathcal{A}}_x} \mathbb{E}\Big[\int_0^T U(t, \bar{c}_t - z\tilde{w}_t) dt \Big], \quad x \in \mathbb{R}, z > 0.$$

- The auxiliary dual problem is defined by

$$\bar{v}(y) = \inf_{\bar{\mathbb{Q}} \in \widetilde{\mathcal{D}}(y)} \mathbb{E}\Big[\int_0^T V(t, \Gamma_t^{\bar{\mathbb{Q}}}) dt \Big] - \langle z\tilde{w}, \bar{\mathbb{Q}} \rangle,$$

  and it admits a unique optimal solution $\bar{\mathbb{Q}}^* \in \widetilde{\mathcal{D}}(y)$.

Unconstrained Auxiliary Problems

- Value functions $\bar{u}(x)$ and $\bar{v}(y)$ are conjugate of each other, i.e.,

$$\bar{v}(y) = \sup_{x \in \mathbb{R}}[\bar{u}(x) - xy],$$
$$\bar{u}(x) = \inf_{y>0}[\bar{v}(y) + xy].$$

- The primal unconstrained auxiliary problem admits the unique solution $\bar{c}^*(x)$ and the unique dual optimizer $\bar{\mathbb{Q}}^*(y)$ and the unique primal optimizer $\bar{c}^*(x)$ satisfies

$$\bar{c}_t^*(x) = I(t, \Gamma_t^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_t, \quad \mathbb{P}\text{-a.s.} \ \ \forall t \in [0, T],$$

where $x = -\bar{v}'(y)$ and $I(t, \cdot) = (U')^{-1}(\cdot)$. .

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$$\bar{c}_t^*(x) = I(t, \Gamma_t^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_t, \quad \mathbb{P}\text{-a.s.} \quad \forall t \in [0, T],$$

where $x = -\bar{v}'(y)$ and $I(t,\cdot) = (U')^{-1}(\cdot)$. .

- Choice of $\tilde{\xi}^*$ using unconstrained problems:

Step 1: Construction of the endogenous stopping time

$$\tau^*(y) \triangleq \inf\{t \geq 0 : I(t, \Gamma_t^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_t \geq 0\} \wedge T.$$

Step 2: Prove the following results: for $\tau^*(y) \leq t \leq T$,

$$I(t, \Gamma_t^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_t \geq -\int_{\tau^*(y)}^{t} \delta_s e^{\int_s^t(\delta_v - \alpha_v)dv}\Big(I(s, \Gamma_s^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_s\Big)ds,$$

Constrained Auxiliary Problems

- For each $y > 0$, we will construct the **valid** stochastic Lagrange multiplier $\tilde{\xi}_t^*(y)$ by

$$\tilde{\xi}_t^*(y) \triangleq 0, \quad \tau^*(y) \leq t \leq T,$$

and

$$\tilde{\xi}_t^*(y) \triangleq \Gamma_t^{\bar{\mathbb{Q}}^*}(y) - U'(t, -z\tilde{w}_t), \quad 0 \leq t \leq \tau^*(y),$$

which implies that

$$I(t, -\tilde{\xi}_t^*(y) + \Gamma_t^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_t = 0, \quad 0 \leq t \leq \tau^*(y).$$

- Let us define the dual value function

$$\tilde{v}(y) \triangleq v(y, \tilde{\xi}^*(y)),$$

the conjugate duality between value functions $\tilde{u}(x)$ and $\tilde{v}(y)$ holds,

$$\tilde{u}(x) = \inf_{y>0}(\tilde{v}(y) + xy),$$
$$\tilde{v}(y) = \sup_{x>0}(\tilde{u}(x) - xy).$$

Constrained Auxiliary Problems

- For the choice of $\tilde{\xi}_t^*$, we can show the existence of the dual optimizer $\widetilde{\mathbb{Q}}^*(y, \tilde{\xi}_t^*)$ (short as $\widetilde{\mathbb{Q}}^*(y)$) for the constrained dual problem and $\widetilde{\mathbb{Q}}^*(y) = \bar{\mathbb{Q}}^*(y)$.

- For each initial wealth $x > 0$, the optimal auxiliary solution $\tilde{c}_t^*(x)$ satisfies

$$\begin{aligned}\tilde{c}_t^*(x) &= I(t, -\tilde{\xi}_t^*(y) + \Gamma_t^{\widetilde{\mathbb{Q}}^*}(y)) + z\tilde{w}_t \\ &= I(t, -\tilde{\xi}_t^*(y) + \Gamma_t^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_t, \quad 0 \le t \le T.\end{aligned}$$

  and

$$\begin{aligned}\tilde{c}_t^*(x) &= 0, \quad 0 \le t \le \tau^*(y), \\ \tilde{c}_t^*(x) &= \bar{c}_t^*(\bar{x}), \quad \tau^*(y) \le t \le T,\end{aligned}$$

  where $y = \tilde{u}'(x)$ and $\bar{x} = -\bar{v}'(y)$ and $\bar{c}^*(\bar{x})$ is the optimal solution for the unconstrained problem starting with the initial value $\bar{x}$.

Constrained Auxiliary Problems

- For each $x > 0$, the optimal consumption $c_t^*(x)$ to the primal utility maximization problem exists and is unique and

$$c_t^*(x) = \tilde{c}_t^*(x) + \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v)dv} \tilde{c}_s^*(x)ds, \quad 0 \le t \le T.$$

In particular,

$$c_t^*(x) = 0, \quad 0 \le t \le \tau^*(y),$$

$$c_t^*(x) = \bar{c}_t^*(\bar{x}) + \int_{\tau^*(y)}^t \delta_s e^{\int_s^t (\delta_v - \alpha_v)dv} \bar{c}_s^*(\bar{x})ds, \quad \tau^*(y) \le t \le T,$$

where $y = \tilde{u}'(x)$ and $\bar{x} = -\bar{v}'(y)$ and $\bar{c}^*(\bar{x})$ is the optimal solution for the unconstrained auxiliary problem.

Constrained Auxiliary Problems

- For the unconstrained auxiliary primal optimizer $\bar{c}^*$, let us go back to the original market model and

$$\hat{c}_t^* \triangleq \bar{c}_t^* + \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \bar{c}_s^* \, ds,$$

corresponds to the unconstrained optimal consumption process.

- For each $x > 0$, the optimal consumption process has the special structure that $c_t^*(x) = 0$ for $0 \le t \le \tau^*(y)$ and

$$c_t^*(x) = \hat{c}_t^*(\bar{x}) - \int_0^{\tau^*(y)} \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \left(\hat{c}_s^*(\bar{x}) - \int_0^s \delta_u e^{-\int_u^s \alpha_v dv} \hat{c}_u^*(\bar{x}) du \right) ds$$

for $\tau^*(y) \le t \le T$ where $y = \tilde{u}'(x)$ and $\bar{x} = -\bar{v}'(y)$.

Future Work

- More explicit structures on the optimal consumption in concrete market models. Graphic comparison on the special structures.
- Other types of non-addictive habit formation or nonlinear habit formations.
- Optimal contract theory when the agent follows the habit formation preference.
- Market Equilibrium under non-addictive habit formation (and/or addictive habit formation).

Thank you for the attention!

Correct Functional Space

- To be precise, $\widetilde{\mathcal{D}}$ is defined as the bipolar set of $\widetilde{\mathcal{M}}$, i.e.,

$$(\widetilde{\mathcal{M}})^\circ \triangleq \left\{ \tilde{c} \in \mathbb{L}^\infty(\mathcal{O}, \widetilde{\mathbb{P}}) : \ \left\langle \tilde{c}, \widetilde{\mathbb{Q}} \right\rangle \leq 1, \ \text{for all } \widetilde{\mathbb{Q}} \in \widetilde{\mathcal{M}} \right\},$$

$$(\widetilde{\mathcal{D}}) \triangleq \left\{ \widetilde{\mathbb{Q}} \in \mathbf{ba}(\mathcal{O}, \widetilde{\mathbb{P}}) : \ \left\langle \tilde{c}, \widetilde{\mathbb{Q}} \right\rangle \leq 1, \ \text{for all } \tilde{c} \in (\widetilde{\mathcal{M}})^\circ \right\}.$$

- However, financially speaking, the process $\tilde{c} \in \widetilde{\mathcal{A}}_x$ is not necessarily in $\mathbb{L}^\infty$!

Correct Functional Space

- Consider a subspace of $\mathbb{L}^0$ denoted by $\mathcal{V}^{\widetilde{\mathcal{M}}}$ whose elements satisfy

$$\| \tilde{c} \|_{\widetilde{\mathcal{M}}} < \infty, \quad \text{where} \quad \| \tilde{c} \|_{\widetilde{\mathcal{M}}} \triangleq \sup_{\widetilde{\mathbb{Q}} \in \widetilde{\mathcal{M}}} \langle |\tilde{c}|, \widetilde{\mathbb{Q}} \rangle.$$

It is clear that $\| \cdot \|_{\widetilde{\mathcal{M}}}$ defines a norm on $\mathcal{V}^{\widetilde{\mathcal{M}}}$, moreover, one can prove that $(\mathcal{V}^{\widetilde{\mathcal{M}}}, \| \cdot \|_{\widetilde{\mathcal{M}}})$ is a Banach space.

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- At this point, for each $\tilde{c} \in \mathcal{V}^{\widetilde{\mathcal{M}}}$ and constant $y > 0$ and $\widetilde{\mathbb{Q}} \in \widetilde{\mathcal{D}}(y)$, we define

$$\langle \tilde{c}, \widetilde{\mathbb{Q}} \rangle \triangleq \sup \left\{ \langle \tilde{c}', \widetilde{\mathbb{Q}} \rangle : \ \tilde{c}' \in \mathbb{L}^\infty, \ \tilde{c}' \leq \tilde{c} \right\}.$$

  Then $\langle \tilde{c}, \widetilde{\mathbb{Q}} \rangle \leq y \| \tilde{c} \|_{\widetilde{\mathcal{M}}} < \infty$ for any $\widetilde{\mathbb{Q}} \in \widetilde{\mathcal{D}}(y) \triangleq y\widetilde{\mathcal{D}}$. Therefore, it is natural to consider the bilinear form between $\mathcal{V}^{\widetilde{\mathcal{M}}}$ and $\mathbf{ba}^{\widetilde{\mathcal{M}}}$, where $\mathbf{ba}^{\widetilde{\mathcal{M}}}$ is defined as the linear space spanned by $\widetilde{\mathcal{D}}$, i.e.

$$\mathbf{ba}^{\widetilde{\mathcal{M}}} \triangleq \left\{ \widetilde{\mathbb{Q}} \in \mathbf{ba}(\mathcal{O}, \widetilde{\mathbb{P}}) : \ \exists y > 0, \mathbb{Q}^+, \mathbb{Q}^- \in \widetilde{\mathcal{D}}(y) \text{ such that } \widetilde{\mathbb{Q}} = \mathbb{Q}^+ - \mathbb{Q}^- \right\}.$$

Correct Functional Space

- The dual norm can be defined as

$$\| \widetilde{\mathbb{Q}} \|_{\mathbf{ba}^{\widetilde{\mathcal{M}}}} \triangleq \sup_{\tilde{c} \in \mathcal{V}^{\widetilde{\mathcal{M}}} : \|\tilde{c}\|_{\widetilde{\mathcal{M}}} \leq 1} | \langle \tilde{c}, \widetilde{\mathbb{Q}} \rangle | .$$

- We can identify $\mathbf{ba}^{\widetilde{\mathcal{M}}}$ as the topological dual of $\mathcal{V}^{\widetilde{\mathcal{M}}}$ and $\widetilde{\mathcal{D}}(y)$ becomes its bounded subset. Moreover, the set $\widetilde{\mathcal{D}}(y)$ is closed in the $\sigma(\mathbf{ba}^{\widetilde{\mathcal{M}}}, \mathcal{V}^{\widetilde{\mathcal{M}}})$-topology, actually, one can show that $\widetilde{\mathcal{D}}(y)$ is $\sigma(\mathbf{ba}^{\widetilde{\mathcal{M}}}, \mathcal{V}^{\widetilde{\mathcal{M}}})$-compact.