On upper functions of solutions to linear SDE's with non-exponentially stable state matrix

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Upper functions of stochastic processes

Let $Y_t,\ t \geq 0,$ be a scalar stochastic process defined on a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$.

Next we adopt a slightly modified definition of upper function from [GikhSkor, p. 289]:

Definition 1

A non-random function $h_t > 0,\ t \geq 0,$ is called an upper function of Y_t if

$$\limsup_{t\to\infty} \frac{Y_t}{h_t} < \infty \quad \text{holds with probability } 1\,. \tag{1}$$

From (1) $\Rightarrow$ there exist a constant $c_0 > 0$ and an a.s. finite moment t_0:

$$Y_t \leq c_0 h_t \quad \text{a.s. for any } t > t_0.$$

Example

Law of iterated logarithm for Brownian motion $\Rightarrow$

$h_t = \sqrt{t \ln \ln t}$ for $Y_t = \|w_t\|$

(w_t is a multidimensional BM, $\|\cdot\|$ is the Euclidean norm)

Upper functions

– provide a.s. deterministic upper bounds on stochastic processes (asymptotically)

if $h_t \to 0, \ t \to \infty,$ then $\limsup\limits_{t\to\infty} Y_t \leq 0$ a.s.

– are used to obtain normalizing functions $g_t : \limsup\limits_{t\to\infty}(Y_t/g_t) \leq 0$ a.s.

$g_t > 0, t \geq 0,$ is any function such that

$\lim\limits_{t\to\infty}(h_t/g_t) = 0$ holds.

In this work we derive upper functions of $Y_t = \|X_t\|^2,$ where $X_t, t \geq 0,$ is a solution to linear SDE with non-exponentially stable state matrix.

Set up

Let $X_t, t \geq 0,$ be an n-dimensional stochastic process defined on a stochastic basic by

$$dX_t = A_t X_t \, dt + G_t \, dw_t \,, \quad X_0 = x \,,$$

where A_t, G_t are known non-random matrices, $\int\limits_0^\infty \|G_t\|^2 \, dt > 0$;

$w_t, t \geq 0,$ – is a d-dimensional Brownian motion;

x is a non-random vector.

We will assume that the matrix A_t is non-exponentially stable, characterizing this property by a stability rate.

Let $\Phi(t,s)$ be the fundamental matrix corresponding to A_t : a solution to

$$\frac{\partial\Phi(t,s)}{\partial t} = A_t\Phi(t,s), \quad \frac{\partial\Phi(t,s)}{\partial s} = -\Phi(t,s)A_s, \quad \Phi(t,t) = \Phi(s,s) = I\,,$$

where I – identity matrix.

Definition 2

We say that A_t is stable with the rate δ_t or δ_t–stable if

(i) $\delta_t > 0, t \geq 0$ and $\limsup\limits_{t\to\infty}(\|A_t\|/\delta_t) < \infty$;

(ii) there exists a constant $\kappa > 0$ such that

$$\|\Phi(t,s)\| \leq \kappa \exp\{-\int_s^t \delta_v\,dv\}\,, \qquad s \leq t\,;$$

(iii) $\int_0^t \delta_s\,ds \to \infty,\ t \to \infty\,.$

Comments

(i) $\delta_t > 0, t \geq 0$ and $\limsup_{t\to\infty}(\|A_t\|/\delta_t) < \infty \Rightarrow \|A_t\| \leq \hat{\kappa}\delta_t,\ t > \hat{t}_0$,
where $\hat{\kappa}$ is a constant:

δ_t is the best possible stability rate available for A_t (up to a scaling factor, i.e. $\tilde{\delta}_t = \lambda\delta_t,\ \lambda > 0$),

as we see from the lower-bound Lyapunov estimate

$$\|\Phi(t,s)\| \geq \bar{\kappa}\exp\left\{-\int_s^t \|A_v\|\,dv\right\}, \qquad s \leq t$$

valid for some constant $\bar{\kappa} > 0$.

Comments

(ii)

$$\|\Phi(t,s)\| \leq \kappa \exp\{-\int\limits_s^t \delta_v\, dv\}, \qquad s \leq t$$

for some constant $\kappa > 0$

If $\delta_t \equiv \kappa_1 > 0 \Rightarrow$ usual exponential stability

If $\delta_t \to 0,\ t \to \infty \Rightarrow$ sub-exponential stability (weaker that the exponential)

If $\delta_t \to \infty,\ t \to \infty \Rightarrow$ super-exponential stability (stronger that the exponential)

The notions of sub- and super- (non-exponential) stability were initially developed for nonlinear differential equations in relation with the Lyapunov exponents, see [Car].

Comments

(iii) $\int_0^t \delta_s \, ds \to \infty, \; t \to \infty$

(iii) $\Rightarrow \|\Phi(t,s)\| \to 0\,, \; t \to \infty,$ being a standard condition defining the asymptotic stability of solutions to linear differential equations:

$$dx_t = A_t x_t \, dt \quad x_s = x \;\Rightarrow\; x_t = \Phi(t,s)x$$

From (ii)-(iii) $\Rightarrow \delta_t$ defines the rate of stability, so it's natural to say that A_t is stable with the rate δ_t.

Motivation

Why studying linear SDEs with non-exponentially stable state matrix?

Why upper functions are of significant importance?

- various applications in modeling
  - physics (anomalous diffusions) [SafCher];
  - reliability theory and engineering (deterioration and damage) [BhaEll],[DeBar];
  - climatology (ice extent and temperature evolution) [Kwasn],[ZapAl];
  - finance (interest rates, inflation) [LiPel],[LeiWu];
  - cognitive science (evidence accumulation) [SmiRat];
- stochastic control theory (non-standard LQG control)

Previous work

Exponentially stable A_t and bounded G_t only

- $A_t = -I$; limiting sets of X_t, conditions on G_t to have $X_t \to 0$, $t \to \infty$ a.s. [Bal]
- scalar case: $A_t \equiv -1$, fading perturbations $G_t \to 0, \; t \to \infty$; asymptotic behavior of X_t [AppCheRod]
- $h_t = \ln t$ [BelKaPr]
- $h_t = \sup_{t \le s}(\alpha_s \ln s)$ [BelPal]

$$\alpha_t = e^{-2\kappa_1 t} \int_0^t e^{2\kappa_1 s} \|G_s\|^2 \, ds \,, \quad \kappa_1 > 0 \text{ is the stability rate of } A_t$$

A key approach: the law of iterated logarithm for time-changed BM.

Assumptions and main result

Assumption $\mathcal{AG}$

the state matrix A_t is stable with the rate δ_t;

the diffusion matrix G_t is such that $\limsup\limits_{t\to\infty} (\|G_t\|/\delta_t) < \infty$.

Remark If Assumption $\mathcal{AG}$ holds $\Rightarrow$ $E\|X_t\|^2$ is bounded, $t \geq 0$.

For a given $0 < \gamma < 1/2$ define

$$d_t = \int_0^t \exp\{-2\gamma \int_s^t \delta_v \, dv\} \|G_s\|^2 \, ds \, . \tag{2}$$

Then d_t is also bounded, $t \geq 0$.

Theorem 1

Let Assumption $\mathcal{AG}$ hold. Then h_t for $Y_t = \|X_t\|^2$ is given by

a) for $d_t = \int\limits_0^t \exp\{-2\gamma \int\limits_s^t \delta_v\, dv\} \|G_s\|^2\, ds$:

$$h_t = d_t \ln\left(\int\limits_0^t \delta_v\, dv\right), \tag{3}$$

if $\int\limits_0^t \exp\{2\gamma \int\limits_0^s \delta_v\, dv\} \|G_s\|^2\, ds \to \infty,\ t \to \infty$;

b)

$$h_t = \exp\{-2\alpha\gamma \int\limits_0^t \delta_v\, dv\}, \tag{4}$$

if $\int\limits_0^\infty \exp\{2\gamma \int\limits_0^s \delta_v\, dv\} \|G_s\|^2\, ds < \infty$

where α, γ are some constants: $0 < \alpha < 1, \quad 0 < \gamma < 1/2$.

Idea of the proof

- Consider a simple linear SDE

$$d\hat{X}_t = -\hat{\delta}_t \hat{X}_t dt + G_t dw_t\,, \qquad \hat{X}_0 = 0\,,$$

where $\limsup_{t\to\infty} (\hat{\delta}_t/\delta_t) < \infty$.

- Obtain an upper function $\hat{h}_t$ of $\|\hat{X}_t\|^2$ from

– the law of iterated logarithm for time-changed BM $w_{\hat{t}}$ if a) condition holds

– non-exponential decay in the b) case.

- Define $Z_t = X_t - \hat{X}_t$ and show that $\sqrt{\hat{h}_t}$ is also an upper function of $\|Z_t\|$ for a proper choice of $\hat{\delta}_t = \gamma\delta_t$.

- Set $h_t = \hat{h}_t$.

Comments and remarks

Since d_t is bounded, the function

$$h_t^{(0)} = \ln(\int_0^t \delta_v \, dv)$$

is the roughest estimate of upper function h_t (an analogy to $h_t^{(0)} = \ln t$).

For some constant $0 < \beta < 1$ the function

$$h_t^{(1)} = \exp\{-\beta \int_0^t \delta_v \, dv\}$$

serves as the lower bound on h_t.

Thus for any upper function from Theorem 1 and some constants $c_1, c_2 > 0$

$$c_2 h_t^{(1)} \leq h_t \leq c_1 h_t^{(0)}$$

Corollary

Let assumptions of Theorem 1 hold and define

$$h_t^{(0)} = \ln\left(\int_0^t \delta_v \, dv\right)$$

a) Convergence to zero:

if $\quad (\|G_t\|^2 h_t^{(0)}/\delta_t) \to 0,$ then

$$\|X_t\|^2 \to 0 \quad \text{a.s.}, \; t \to \infty$$

b) Normalizing function:

if $\quad (\|G_t\|^2/\delta_t) \to 0,$ then

$$\|X_t\|^2/h_t^{(0)} \to 0 \quad \text{a.s.}, \; t \to \infty$$

c) Equivalence:

if $\quad \liminf_{t\to\infty} (\|G_t\|^2/\delta_t) > 0,$ then

$$h_t = c_0 h_t^{(0)}, \quad \text{where } c_0 > 0 \text{ is some constant.}$$

Examples of upper functions

We assume that $\liminf_{t\to\infty} (\|G_t\|^2/\delta_t) > 0$ (c) of Corollary)

- Power family: $\delta_t = a(1+t)^b$, $\quad a > 0,\ b > -1$

  $-1 \le b < 0$: sub-exponential stability

  $b = 0$: exponential stability

  $b > 0$: super-exponential stability

  $h_t \sim \ln t$ if $b > -1$ and $h_t \sim \ln\ln t$ if $b = -1$

- Logarithmic family: $\delta_t = a \ln(e+t)^b$, $\quad a > 0,\ b \neq 0$

  $b < 0$: sub-exponential stability

  $b > 0$: super-exponential stability

  $h_t \sim \ln t$

- Exponential family: $\delta_t = a \exp\{t^b\}$, $\quad a > 0,\ b > 0$

  super-exponential stability

  $h_t \sim t^b$

Non-standard infinite time LQG control

Controlled stochastic process $Z_t,\ t \geq 0,$ governed by

$$dZ_t = C_t Z_t dt + B_t U_t dt + G_t dw_t\,, \qquad Z_0 = z, \tag{5}$$

where $U_t, t \geq 0,$ is an admissible control, i.e. an $\mathcal{F}_t = \sigma\{w_s,\ s \leq t\}$–adapted process s.t. there exists a solution to (5);

z is a non-random vector;

$C_t,\ B_t,\ G_t$ are known non-random matrices of appropriate dimensions;

B_t : bounded and $B_t B_t' \geq bI$ for some $b > 0$;

C_t : $\|C_t\| \to \infty,\ t \to \infty$ and C_t is super exponentially anti-stable.

Definition 3

C_t is called anti-stable if $-C_t'$ is stable. If C_t is anti-stable with the rate δ_t:

$$\|\Phi(t,s)\| \geq \tilde{\kappa} \exp\Big\{\int_s^t \delta_v\, dv\Big\}\,, \qquad s \leq t\,.$$

The cost functional J_T over planning horizon $[0, T]$:

$$J_T(U) = \int_0^T \left[Z_t' Q_t Z_t + U_t' U_t \right] dt,$$

where symmetric matrices $Q_t \geq qI$ ($q > 0$ is some constant, $'$ denotes the matrix transpose); $U \in \mathcal{U}$, $\mathcal{U}$ is the set of admissible controls.

Lemma

There exists a symmetric $\Pi_t \geq 0$, $t \geq 0$, satisfying the Riccati equation

$$\dot{\Pi}_t + \Pi_t C_t + C_t' \Pi_t - \Pi_t B_t B_t' \Pi_t + Q_t = 0,$$

such that:

(i) $\alpha \delta_t \leq \Pi_t \leq \beta \delta_t$ for some constants $\alpha, \beta > 0$;

(ii) the matrix $A_t = C_t - B_t B_t' \Pi_t$ is $\tilde{\delta}_t$-super exponentially stable with $\tilde{\delta}_t = \lambda \delta_t$, where δ_t is the anti-stability rate of C_t, $\lambda > 0$ is a constant.

Now we may define the stable feedback

$$U_t^* = -B_t' \Pi_t Z_t^* \tag{6}$$

where Z_t^* , $t \geq 0$, satisfies the linear SDE

$$dZ_t^* = (C_t - B_t B_t' \Pi_t) Z_t^* dt + G_t dw_t \,, \quad Z_0^* = z \,. \tag{7}$$

Assumption $\mathcal{G}$

$$c_G = \limsup_{t \to \infty} (\|G_t\| / \delta_t) < \infty$$

Definition 4

The control $U^* \in \mathcal{U}$ is said to be

average overtaking optimal over an infinite time-horizon if
$$\limsup_{T \to \infty} (EJ_T(U^*) - EJ_T(U)) \leq 0$$

average ϵ-overtaking optimal over an infinite time-horizon if
$$\limsup_{T \to \infty} (EJ_T(U^*) - EJ_T(U)) \leq \epsilon \quad \text{for some } \epsilon > 0$$

and any $U \in \mathcal{U}$.

Theorem 2

Let Assumption $\mathcal{G}$ hold. Then the stable feedback U^* is

average overtaking optimal over an infinite time-horizon if in $\mathcal{G}$: $c_G = 0$

average ϵ-overtaking optimal over an infinite time-horizon if in $\mathcal{G}$: $c_G > 0$.

Pathwise cost comparison

It can be shown that under $\mathcal{G}$, for any $U \in \mathcal{U}$, the costs

$$J_T(U^*) \leq J_T(U) + c\|Z_T^*\|^2 + h_T^* \quad \text{a.s.}, \ T \to \infty,$$

where $c > 0$ is some constant, $h_T^* > 0$ is any non-random function s.t. $h_T^* \to \infty, \ T \to \infty$.

For Z_T^* : all conditions of Theorem 1 are satisfied $\Rightarrow$ upper function h_T $\Rightarrow$ pathwise optimality of U^* for a properly chosen normalizing function.

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Thank you for your attention!