Affine Volterra processes and their characteristic functions

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Rough volatility models

- Empirical studies indicate **volatility is rougher than BM**: Gatheral, Jaisson & Rosenbaum ('14); Bennedsen, Lunde, Pakkanen ('16), ...
- Subsequent development of **stochastic models with this feature**: Gatheral, Jaisson & Rosenbaum ('14); Guennoun, Jacquier & Roome ('14); Bayer, Friz & Gatheral (15); Bennedsen, Lunde, Pakkanen ('16); **El Euch & Rosenbaum ('16,'17)**, ...
- These models are able to
  - match roughness of time series data
  - fit implied volatility smiles remarkably well
  - admit in some cases microstructural justification
- Mathematically, this rests on **fractional Brownian motion** in the tradition of Kolmogorov ('40), Mandelbrot & van Ness ('68), ...

Rough Heston model

- The **Heston model** is the stock price model

$$\frac{dS_t}{S_t} = \sqrt{X_t}d\widetilde{W}_t$$

where the volatility follows a CIR process,

$$X_t = X_0 + \int_0^t \kappa(\theta - X_s)ds + \int_0^t \sigma\sqrt{X_s}dW_s$$

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- **El Euch & Rosenbaum ('16)** study the **rough Heston model** obtained by replacing the CIR process by the **rough CIR process**

$$X_t = X_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\kappa(\theta - X_s)ds + \sigma\sqrt{X_s}dW_s\Big)$$

where $\alpha \in (\frac{1}{2}, 1)$.

Rough Heston model

- Inspired by the Riemann–Liouville fractional Brownian motion introduced by Lévy ('53)
- Hölder continuous paths of any order less than $H = \alpha - \frac{1}{2}$
- Microstructural foundation as scaling limit of Hawkes processes

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But:

- Existence and uniqueness is non-trivial: El Euch & Rosenbaum construct the rough CIR using Hawkes processes.
- Not a semimartingale, not Markovian ...
- ... so how to price options? This is needed for implied volatilities!

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Warmup: The standard Heston model

Characteristic function of the (standard) Heston model

- The Heston model is tractable because $(\log S_t, X_t)$ **is affine**. This gives **explicit characteristic function**:

$$\mathbb{E}[e^{u \log S_T}] = e^{\phi(T)+\psi(T)X_0}$$

for $u \in \mathrm{i}\mathbb{R}$ and $S_0 = 1$, where (ϕ, ψ) solves the **Riccati equations**

$$\begin{aligned} \phi' &= \kappa\theta\psi & \phi(0) &= 0 \\ \psi' &= \frac{1}{2}(u^2 - u) - (u\rho\sigma - \kappa)\psi + \frac{\sigma^2}{2}\psi^2 & \psi(0) &= 0 \end{aligned}$$

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$$\frac{dM_t}{M_t} = -\left\{ \left(\phi' - \kappa\theta\psi\right) + \left(\psi' - \left[\frac{1}{2}(u^2 - u) - \cdots\right]\right) X_t \right\} dt + (dW_t \text{ term})$$

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Hence, provided M is a martingale (which follows from $\operatorname{Re}\psi \le 0$),

$$\mathbb{E}[e^{u \log S_T}] = \mathbb{E}[M_T] = M_0 = e^{\phi(T)+\psi(T)X_0}.$$

What about the rough Heston model?

- Remarkably, El Euch & Rosenbaum obtain an analogous result for the rough Heston model.
- **Notation:** $D^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s) ds$

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Theorem. (El Euch & Rosenbaum, '16) Assume we are given a solution ψ of the **"fractional Riccati equation"**

$$D^\alpha \psi = \frac{1}{2}(u^2 - u) + (u\rho\sigma - \kappa)\psi + \frac{\sigma^2}{2}\psi^2$$

and define ϕ and χ by

$$\phi' = \kappa\theta\chi, \quad \phi(0) = 0; \qquad \chi' = D^\alpha \psi, \quad \chi(0) = 0.$$

Then

$$\mathbb{E}[e^{u \log S_T}] = e^{\phi(T) + \chi(T) X_0}$$

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Proof: Rather involved. Uses the Hawkes approximation.

Questions

- Can the proof of this result be simplified?
- Can the Hawkes approximation be avoided?
- What about the joint characteristic function of $(\log S_T, X_T)$?
- What about conditional characteristic function?
- More general specifications?

Affine Volterra processes

- State space $E \subseteq \mathbb{R}^d$
- Affine diffusion and drift coefficients

$$
\begin{aligned}
a(x) &= A^0 + A^1 x_1 + \cdots + A^d x_d \\
b(x) &= b^0 + b^1 x_1 + \cdots + b^d x_d
\end{aligned}
$$

  with $A^i \in \mathbb{S}^d$, $b^i \in \mathbb{R}^d$, and $a(x) \succeq 0$ on E.
- $\sigma \colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ continuous with $\sigma(x)\sigma(x)^\top = a(x)$ on E.
- Matrix-valued kernel $K \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{d\times d})$.

A continuous E-valued solution X of the stochastic Volterra equation

$$
X_t = X_0 + \int_0^t K(t-s) b(X_s) ds + \int_0^t K(t-s)\sigma(X_s) dW_s
$$

is called an **affine Volterra process** (of convolution type).

Affine Volterra processes

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- **Example:** For usual affine diffusions, take $K(t) \equiv \mathrm{id}$.
- **Example:** The volatility process in the rough volatility model by Rosenbaum & El Euch is obtained with

$$K(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$$

- **Example:** More generally, the full rough volatility model uses $d = 2$ and the kernel

$$K(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \end{pmatrix}$$

Characteristic function

Notation: $(F * G)(t) = \int_0^t F(t-s)G(s)ds$.

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Theorem. Let $d = 1$. Fix $u \in \mathbb{C}$, assume $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C})$ solves

$$\psi = uK + \left(\psi b^1 + \frac{A^1}{2}\psi^2\right) * K, \tag{1}$$

and define ϕ and χ by $\phi(0) = 0$, $\chi(0) = u$, and

$$\phi' = \psi b^0 + \frac{A^0}{2}\psi^2, \qquad \chi' = \psi b^1 + \frac{A^1}{2}\psi^2. \tag{2}$$

Then, under a martingale condition,

$$\mathbb{E}[e^{uX_T}] = e^{\phi(T)+\chi(T)X_0}, \qquad T \geq 0.$$

Characteristic function

Proof.

- **Ansatz:** Fix T and consider the semimartingale $M_t = e^{Y_t}$, where

$$Y_t = \phi(T-t) + \chi(T)X_0 - \int_0^t \chi'(T-s)X_s ds + \int_0^t \psi(T-s)dZ_s$$

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- **Itô** yields

$$\frac{dM_t}{M_t} = \left(-\phi' + \psi b^0 + \frac{A^0}{2}\psi^2\right)dt + \left(-\chi' + \psi b^1 + \frac{A^1}{2}\psi^2\right)X_t dt + (dW_t \text{ term}),$$

a local martingale **by (2)**. By "martingale condition", a martingale.

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- **(1) and stochastic Fubini** yield $Y_T = \phi(0) + \chi(0)X_T$.
- Hence $\mathbb{E}[e^{uX_T}] = \mathbb{E}[M_T] = M_0 = e^{\phi(T)+\chi(T)X_0}$

Characteristic function

- The above theorem is **probably the quickest way to derive the characteristic function** of X_t ...
- ... although the **"martingale condition"** needed to make the derivation rigorous requires additional work.
- However, the computations do not generalize to **conditional characteristic functions**, which is a nontrivial extension due to lack of Markovian structure.
- Different, more general, and more informative results are possible.

Conditional expectation

- Set $B = [b^1 \ b^2 \ \cdots \ b^d] \in \mathbb{R}^{d \times d}$ and let temporarily $b^0 = 0$:

$$X_t = X_0 + \int_0^t K(t-s) B X_s ds + \int_0^t K(t-s)\sigma(X_s) dW_s$$

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- The kernel $-KB$ admits a **resolvent** $R_B \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{d \times d})$:

$$(KB) * R_B = R_B * (KB) = KB + R_B$$

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- **Example:** If $K \equiv \text{id}$ then $R_B(t) = -Be^{tB}$.
- **Example:** If $K(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$ and $B = -\kappa < 0$, then

$$R_B = f^{\alpha,\kappa}$$

  is the so-called Mittag-Leffler density function.

Conditional expectation

Theorem. The conditional mean, or **forward process**, is given by

$$\mathbb{E}[X_T \mid \mathcal{F}_t] = \left(\mathrm{id} - \int_0^T R_B(s)ds\right) X_0 + \left(\int_0^T E_B(s)ds\right) b^0 + \int_0^t E_B(T-s)\sigma(X_s)dW_s,$$

where R_B is the resolvent of $-KB$ and $E_B = K - R_B * K$.

Conditional characteristic function

- **Recap:**

$$X_t = X_0 + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s$$

with

$$a(x) = A^0 + A^1 x_1 + \cdots + A^d x_d = \sigma(x)\sigma(x)^\top$$
$$b(x) = b^0 + Bx$$

- **Notation:** For any row vector $u \in (\mathbb{C}^d)^*$ we define the row vector

$$A(u) = (uA^1 u^\top, \ldots, uA^d u^\top).$$

- **Riccati–Volterra equation:** $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+, (\mathbb{C}^d)^*)$ such that

$$\psi = uK + \left(\psi B + \frac{1}{2}A(\psi)\right) * K.$$

Conditional characteristic function

Theorem (∗). Let $u \in (\mathbb{C}^d)^*$ and assume the Riccati–Volterra equation has solution ψ. Fix $T < \infty$ and define

$$dY_t = \psi(T-t)\sigma(X_t)dW_t - \frac{1}{2}\psi(T-t)a(X_t)\psi(T-t)^\top dt,$$
$$Y_0 = uX_0 + \int_0^T \left(\psi(s)b(X_0) + \frac{1}{2}\psi(s)a(X_0)\psi(s)^\top\right) ds.$$

Then for $t \leq T$,

$$Y_t = \mathbb{E}[uX_T \mid \mathcal{F}_t] + \frac{1}{2}\int_t^T \psi(T-s)a(\mathbb{E}[X_s \mid \mathcal{F}_t])\psi(T-s)^\top ds.$$

Consequently $\{\exp(Y_t),\, 0 \leq t \leq T\}$ is a local martingale and, if it is a true martingale, one has the affine transform formula

$$\mathbb{E}[e^{uX_T} \mid \mathcal{F}_t] = e^{Y_t}, \quad t \leq T.$$

Conditional characteristic function

Remark. With the same method we actually get a formula for

$$\mathbb{E}\left[\exp\left(uX_T + (f * X)_T\right) \;\middle|\; \mathcal{F}_t\right]$$

for $u \in (\mathbb{C}^d)^*$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, (\mathbb{C}^d)^*)$.

Volterra–Ornstein–Uhlenbeck process

- With $E = \mathbb{R}^d$ and $\sigma(x) \equiv \sigma$ constant we obtain

$$X_t = X_0 + \int_0^t K(t-s)(b^0 + BX_s)ds + \int_0^t K(t-s)\sigma dW_s$$

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  where R_B is the resolvent of $-KB$ and $E_B = K - R_B * K$.
- The quadratic variation of the process Y is deterministic,

$$\langle Y \rangle_t = \int_0^t \psi(T-s)\sigma\sigma^\top \psi(T-s)^\top ds.$$

  Thus the martingale condition holds.

The martingale condition

How to verify the martingale property of e^{Y_t} more generally?

- In the **classical case** $(K \equiv \mathrm{id})$ the condition is

$$\mathrm{Re}\,\phi(t) + \mathrm{Re}\,\psi(t)x \le 0, \quad t \ge 0, \quad x \in E.$$

- Then

$$\mathrm{Re}\,Y_t = \mathrm{Re}\,\phi(t) + \mathrm{Re}\,\psi(t)X_t \le 0,$$

  so that e^{Y_t} is bounded.

The martingale condition

- A **resolvent of the first kind** of K is a kernel L such that

$$K * L = L * K \equiv \mathrm{id}$$

In general L is a measure, for instance $L(dt) = \delta_0(dt)$ if $K \equiv \mathrm{id}$.

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- **Example:** If K is **completely monotone**, then L exists and is the sum of a point mass in zero and a completely monotone function.

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Shift operator: $\Delta_\tau f(t) = f(t+\tau)$

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Lemma. Consider the setting of Theorem (∗), and assume K admits a resolvent of the first kind L. Define

$$\pi_\tau = \Delta_\tau \psi * L - \Delta_\tau(\psi * L),$$

and assume $\pi_\tau \in BV_{\mathrm{loc}}$ for every $\tau \geq 0$. Then

$$Y_t = \phi(\tau) + (\Delta_\tau \psi * L)(0) X_t - \pi_\tau(t) X_0 + (d\pi_\tau * X)_t,$$

with $\tau = T - t$ and

$$\phi(\tau) = \int_0^\tau \left(\psi(s) b^0 + \frac{1}{2} \psi(s) A^0 \psi(s)^\top \right) ds.$$

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In particular: For $E = \mathbb{R}^d_+$, verify martingale condition by controlling the signs of the real parts of $(\Delta_\tau \psi * L)(0)$, $\pi_\tau(t)$, and $d\pi_\tau$.

Volterra square-root process

- $E = \mathbb{R}_+^d$ and diagonal kernel $K = \mathrm{diag}(K_1, \ldots, K_d)$:

$$X_{i,t} = X_{i,0} + \int_0^t K_i(t-s) b_i(X_s) ds + \int_0^t K_i(t-s) \sigma_i \sqrt{X_{i,s}} dW_{i,s}$$

- Inward-pointing drift condition:

$$b^0 \in \mathbb{R}_+^d \text{ and } B_{ij} \geq 0 \text{ for } i \neq j.$$

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- Inward-pointing drift condition:

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- **Assumption:** Each K_i is **completely monotone** and is controlled near zero in the sense that

$$K_i(t) = o(t^{-\gamma_i}) \text{ as } t \to 0$$

for some $\gamma_i < 1/2$ (these assumptions can be relaxed).

The multi-factor Volterra CIR process

Theorem.

- The stochastic Volterra equation has a unique in law $\mathbb{R}_+^d$-valued weak solution for any initial condition $X_0 \in \mathbb{R}_+^d$. The paths of X_i are Hölder continuous of any order less than $H_i = 1/2 - \gamma_i$, for each $i = 1, \ldots, d$.
- For any $u \in (\mathbb{C}^d)^*$ with $\operatorname{Re} u_i \leq 0$ for each $i = 1, \ldots, d$, the Riccati–Volterra equation

$$\psi_i(t) = u_i K_i(t) + \int_0^t K_i(t-s) \left(\psi(s) b^i + \frac{\sigma_i^2}{2} \psi_i(s)^2 \right) ds$$

  has a unique global solution $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+, (\mathbb{C}^d)^*)$.
- The martingale condition in Theorem (∗) holds, as does the affine transform formula.

Conclusion

- Brownian paths are too smooth for volatility modeling
- Affine Volterra processes generalize various known rough volatility models
- Despite lack of Markov property, affine transform formulas can be derived
- Lots to do:
  - Numerical methods for the Riccati–Volterra equations ...
  - ... or even explicit solutions in special cases?
  - Hedging and optimal investment in these models
  - Boundary attainment for Volterra square-root processes
  - Non-convolution kernels $K(t, s)$
  - Etc.

Thank you!