# Bank monitoring incentives under adverse selection

Chao ZHOU

Department of Mathematics, National University of Singapore (NUS)

with Nicolás Hernández and Dylan Possamaï

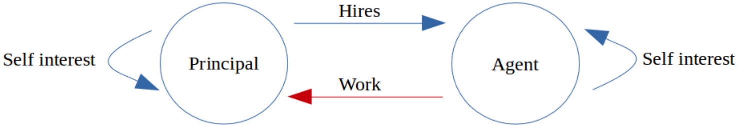
Universidad de Chile & Université Paris-Dauphine

Conference "Asymptotic Statistics of Stochastic Processes and Applications"

17-21 July 2017, Peterhof, Russia

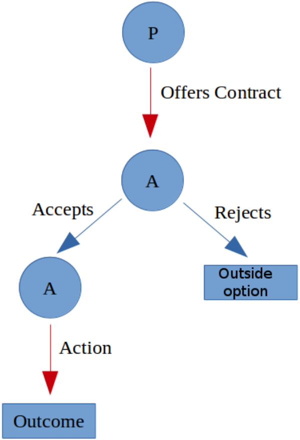
# Outline

The **Principal**-**Agent** problem arises when one person or entity (the **Principal**) hires another one (the **Agent**) to work on behalf of him.




Hidden action: The work/action performed by the **Agent** is **not observable** by the **Principal**.

From the game theory point of view, the **Principal** and the **Agent** play a Non-zero sum Stackelberg game.



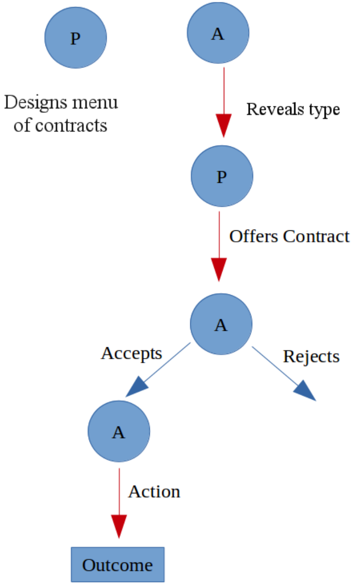
Examples of **Principal**-**Agent** situations:

- **Boss**-**Employee**.
- Agency problems: **Shareholders**-**Corporate management**.
- Electricity markets: **Planner**-**Consumers**.
- Pollution: **Regulator**-**Companies**.

In the literature three main types of Principal-Agent problems are studied

1. First-best (risk sharing).
2. Second-best (moral hazard).
3. Third-best (adverse selection).

# Interaction under adverse selection.



- A **bank** monitors a pool of $I$ identical loans indexed by $j = 1, \ldots, I$.
- Each loan yields cash flow $\mu$ per unit of time until it defaults.
- The **bank** raise funds from an **investor**.
- There are two types of **banks** in the market: the **"good"** bank $\rho_g$ and the **"bad"** bank $\rho_b$. The **investor** does not know the type of the **bank**, only the proportions $p_g$ and $p_b$.

- Denote by $N_t := \sum_{j=1}^{I} \mathbf{1}_{\{\tau^j \leq t\}}$, the current size of the pool at time $t$ is $I - N_t$.
- The action of the **bank** of type $\rho_i$ is to decide how many loans he will **not monitor**

$$k_t^i \in \{0, \ldots, I - N_t\}.$$

- Every non-monitored loan renders a private **benefit** $B$ to the **bank**.
- The **individual default intensity** for each loan is given by

$$\alpha_t^{j,i} := \alpha_{I-N_t}\left(1 + \left(1 - e_t^{j,i}\right)\varepsilon\right).$$

- The associated **aggregated default intensity** is given by

$$\lambda_t^{k^i} := \sum_{j=1}^{I-N_t} \alpha_t^{j,i} = \alpha_{I-N_t}\left(I - N_t + \varepsilon k_t^i\right).$$

The **bank** controls the **distribution** of $N_t$.

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which $N$ is a Poisson process with intensity $\lambda_t^0$.

We denote $\tau$ the **liquidation time** of the whole pool and let $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ be the minimal filtration containing $(\mathcal{F}_t^N)_{t\geq 0}$ and that makes $\tau$ a $\mathbb{G}$-stopping time.

Define $\mathbb{P}^k$ on $\mathcal{G}_t$ by

$$\frac{d\mathbb{P}^k}{d\mathbb{P}} = Z_t^k,$$

where $Z^k$ is the unique solution of the following SDE

$$Z_t^k = 1 + \int_0^t Z_{s-}^k \left( \frac{\lambda_s^k}{\lambda_s^0} - 1 \right) \left( dN_s - \lambda_s^0 ds \right), \; 0 \leq t \leq \tau, \; \mathbb{P} - a.s.$$

Then $N_t - \int_0^t \lambda_t^k ds$, is a $\mathbb{P}^k-$martingale.

The **investor** designs a menu of contracts $(\Psi_i)_{i\in\{g,b\}} := (k^i, \theta^i, D^i)_{i\in\{g,b\}}$ consisting in:

- Predictable, non-decreasing payments $D^i$.
- Probabilities $(1-\theta^i)$ under which the project is liquidated **given a default**.
- Recommended level of effort $k^i$.

Denote $H_t := \mathbf{1}_{t\geq\tau}$, then

$$dH_t = \begin{cases} 0 \text{ with probability } \theta^i_t, \\ dN_t \text{ with probability } 1-\theta^i_t. \end{cases}$$

The **investor** designs a menu of contracts $(\Psi_i)_{i\in\{g,b\}} := (k^i, \theta^i, D^i)_{i\in\{g,b\}}$ consisting in:

- Predictable, non-decreasing payments $D^i$.
- Probabilities $(1-\theta^i)$ under which the project is liquidated **given a default**.
- Recommended level of effort $k^i$.

Denote $H_t := \mathbf{1}_{t\geq\tau}$, then

$$dH_t = \begin{cases} 0 \text{ with probability } \theta^i_t, \\ dN_t \text{ with probability } 1-\theta^i_t. \end{cases}$$

Utility of the **bank** of type $\rho_i$

$$u_0^i(k^i, \theta^i, D^i) := \mathbb{E}^{\mathbb{P}^{\boldsymbol{k^i}}} \left[ \int_0^\tau e^{-rs} (\rho_i dD_s^i + Bk_s^i\, ds) \right],$$

Utility of the **investor**

$$v_0\left( (\Psi_i)_{i \in \{g,b\}} \right) := \sum_{i \in \{g,b\}} p_i \mathbb{E}^{\mathbb{P}^{\boldsymbol{k^i}}} \left[ \int_0^\tau (I - N_s)\, \mu ds - dD_s^i \right].$$

**Agent's problem:**

$$\underset{k \in \mathfrak{K}}{\text{maximize}}\; u_0^i(k, \theta^i, D^i)$$

**Principal's problem:**

$$\text{maximize} \quad v_0(\Psi_g, \Psi_b)$$

$$\begin{aligned}
\text{s.t.} \quad & u_0^i(k^i, \theta^i, D^i) \geq R_0, \; i \in \{g, b\}, \\
& u_0^i(k^i, \theta^i, D^i) = \sup_{k \in \mathfrak{K}} u_0^i(k, \theta^i, D^i), \; i \in \{g, b\}, \\
& u_0^i(k^i, \theta^i, D^i) \geq \sup_{k \in \mathfrak{K}} u_0^i(k, \theta^j, D^j), \; i \neq j, \; (i,j) \in \{g, b\}^2.
\end{aligned}$$

Continuation utility approach:

- Discrete time: Spear and Srivastava (1987).
- Continuous time: Sannikov (2008).
- Bank monitoring: Pagès (2013) and Pagès and Possamaï (2014).

Define the continuation utility of the **bank** at time $t \geq 0$

$$u_t^i(k, \theta^i, D^i) := \mathbb{E}^{\mathbb{P}^k}\left[\int_{t\wedge\tau}^{\tau} e^{-r(s-t)}\left(\rho_i dD_s^i + k_s B ds\right)\bigg|\, \mathcal{G}_t\right].$$

Define also the value function of the **bank** for any $t \geq 0$

$$U_t^i(\theta^i, D^i) := \operatorname*{ess\,sup}_{k\in\mathfrak{K}}\ u_t^i(k, \theta^i, D^i).$$

The process $e^{-rt}u_t^i(k,\theta^i,D^i) + \int_0^t e^{-rs}\left(\rho_i dD_s^i + k_s B ds\right)$ is a $\mathbb{P}^k-$martingale.

There exist $\mathbb{G}-$predictable processes $h^{1,i,k}$ and $h^{2,i,k}$ such that

$$
\begin{aligned}
du_t^i(k,\theta^i,D^i) = & \left(ru_t^i(k,D^i,\theta^i) - Bk_t\right)dt - \rho_i dD_t^i - h_t^{1,i,k}\left(dN_t - \lambda_t^k dt\right) \\
& - h_t^{2,i,k}\left(dH_t - (1-\theta_t^i)\lambda_t^k dt\right), \ 0 \le t < \tau, \ \mathbb{P}-a.s.
\end{aligned}
$$

**The continuation utility is a super-solution to a BSDE with jumps.**

Let us then define

$$Y_t^{i,k} := u_t^i(k, \theta^i, D^i),\ Z_t^{i,k} := (h_t^{1,i,k}, h_t^{2,i,k})^\top,\ M_t := (N_t, H_t)^\top,$$
$$\widetilde{M}_t^i := M_t - \int_0^t \lambda_s^0 (1, 1-\theta_s^i)^\top ds,\ K_t^i := \rho_i D_t^i,$$

so that we can rewrite the previous equation as follows $\mathbb{P} - a.s.$,

$$Y_t^{i,k} = 0 - \int_t^\tau f^i(s, k_s, Y_s^{i,k}, Z_s^{i,k}) ds + \int_t^\tau Z_s^{i,k} \cdot d\widetilde{M}_s^i + \int_t^\tau dK_s^i,\ 0 \le t \le \tau,$$

where

$$f^i(t,k,y,z) := ry - Bk + k\alpha_{I-N_t}\varepsilon z \cdot (1, 1-\theta_t^i)^\top.$$

Denote by $(Y^i, Z^i)$ the unique (super–)solution to the following BSDE

$$Y_t^i = 0 - \int_t^\tau g^i(s, Y_s^i, Z_s^i)ds + \int_t^\tau Z_s^i \cdot d\widetilde{M}_s^i + \int_t^\tau dK_s^i,\ 0 \le t \le \tau,\ \mathbb{P} - a.s.,$$

where

$$\boxed{g^i(t,y,z) := \inf_{k \in \{0,\dots,I-N_t\}} f^i(t,k,y,z).}$$

By the comparison theorems, Royer (2008), the value function of the **bank** has the dynamics, for $t \in [0,\tau]$, $\mathbb{P} - a.s.$

$$dU_t^i(\theta^i, D^i) = \left(rU_t^i(\theta^i, D^i) - Bk_t^{\star,i} + \lambda_t^{k^{\star,i}} Z_t^i \cdot (1, 1-\theta_t^i)^\top\right) dt - \rho_i dD_t^i - Z_t^i \cdot d\widetilde{M}_t^i, \quad (1)$$

and the optimal monitoring choice of the **bank** is given by

$$\boxed{k_t^{\star,i} = (I - N_t)\mathbf{1}_{\{Z_t^i \cdot (1,1-\theta_t^i)^\top < b_t\}}.}$$

Denote by $(Y^i, Z^i)$ the unique (super–)solution to the following BSDE

$$Y_t^i = 0 - \int_t^\tau g^i(s, Y_s^i, Z_s^i)ds + \int_t^\tau Z_s^i \cdot d\widetilde{M}_s^i + \int_t^\tau dK_s^i,\ 0 \le t \le \tau,\ \mathbb{P} - a.s.,$$

where

$$\boxed{g^i(t,y,z) := \inf_{k \in \{0,\dots,I-N_t\}} f^i(t,k,y,z).}$$

By the comparison theorems, Royer (2008), the value function of the **bank** has the dynamics, for $t \in [0,\tau]$, $\mathbb{P} - a.s.$

$$dU_t^i(\theta^i, D^i) = \left( rU_t^i(\theta^i, D^i) - Bk_t^{\star,i} + \lambda_t^{k^{\star,i}} Z_t^i \cdot (1, 1-\theta_t^i)^\top \right) dt - \rho_i dD_t^i - Z_t^i \cdot d\widetilde{M}_t^i,\ (1)$$

and the optimal monitoring choice of the **bank** is given by

$$\boxed{k_t^{\star,i} = (I - N_t)\mathbf{1}_{\{Z_t^i \cdot (1,1-\theta_t^i)^\top < b_t\}}.}$$

The value function of the **investor** in the pure moral hazard case is

$$V_t^{\mathrm{pm}}(R_0) := \operatorname*{ess\,sup}_{(D^i,\theta^i,Z^i)\in\mathcal{A}^i(R_0)} \mathbb{E}^{\mathbb{P}^{k^\star,i}}\left[\int_{t\wedge\tau}^{\tau}(I-N_s)\mu ds - dD_s^i \middle| \mathcal{G}_t\right],$$

where the set of admissible contracts $\mathcal{A}^i(R_0)$ is defined by

$$\mathcal{A}^i(R_0) := \left\{(\theta^i, D^i, Z^i), \text{ s.t. } U_0^i(\theta^i, D^i) \geq R_0\right\}.$$

1 state variable $\Longrightarrow$ associated **system** of HJB equations (**ODE's**).

Define the set $\widehat{\mathcal{V}}_j := [Bj/(r+\widehat{\lambda}_j^{SH}), \infty)$.

For **every** admissible contract $(\theta, D) \in \Theta \times \mathcal{D}$ and $t \geq 0$

$$(U_t^b(\theta, D), U_t^g(\theta, D)) \in \widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}.$$

Do all the points from $\widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ correspond to a pair of value functions of the banks under some admissible contract?

Define the set $\widehat{\mathcal{V}}_j := [Bj/(r + \widehat{\lambda}_j^{SH}), \infty)$.

For **every** admissible contract $(\theta, D) \in \Theta \times \mathcal{D}$ and $t \geq 0$

$$(U_t^b(\theta, D), U_t^g(\theta, D)) \in \widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}.$$

**Do all the points from $\widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ correspond to a pair of value functions of the banks under some admissible contract?**

Credible set approach:

- Cvitanić, Wan and Yang (2013).

## Definition

For any time $t \geq 0$, we define the **credible set** $\mathcal{C}_{I-N_t}$ as the set of $(u^b, u^g) \in \widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ such that there exists some admissible contract $(\theta, D) \in \Theta \times \mathcal{D}$ satisfying $U_t^b(\theta, D) = u^b$, $U_t^g(\theta, D) = u^g$ and $(U_s^b(\theta, D), U_s^g(\theta, D)) \in \widehat{\mathcal{V}}_{I-N_s} \times \widehat{\mathcal{V}}_{I-N_s}$ for every $s \in [t, \tau)$, $\mathbb{P} - a.s.$

We denote by $\mathfrak{U}_t(u^b)$ the largest value $U_t^g(\theta, D)$ that the good bank can obtain from all the contracts $(\theta, D)$ under which the value of the bad bank is $u^b$. We also denote the lowest one by $\mathfrak{L}_t(u^b)$.

- $\mathfrak{L}_t(u^b)$ can be simply obtained.
- $\mathfrak{U}_t(u^b)$ solves a stochastic control problem
  $\Longrightarrow$ associated **system** of HJB equations (**ODE's**).

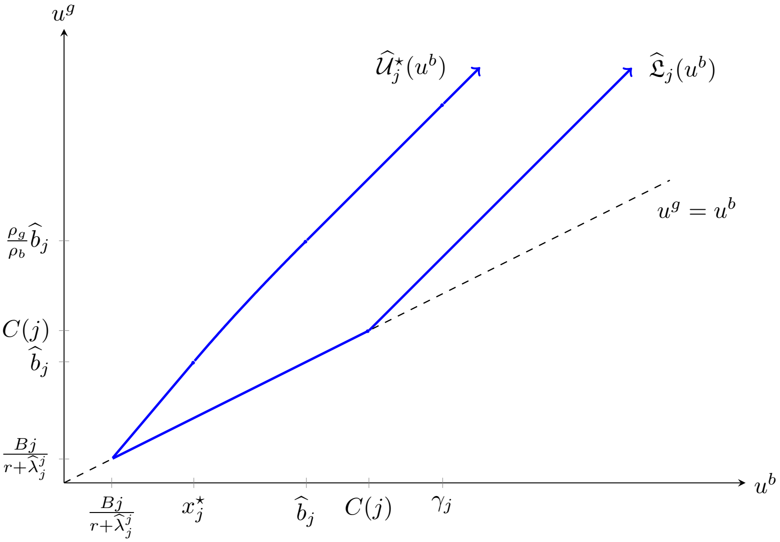



Figure : Credible set with $j$ loans left.

The **investor** designs two contracts:

$$\Psi_g = (D^g, \theta^g), \ \Psi_b = (D^b, \theta^b).$$

Each one of these contracts will depend on two state variables, due to the **temptation processes** of each agent.

Contract designed for the good agent:

$$dU_t^g(\theta^g, D^g) = \left( rU_t^g(\theta^g, D^g) - Bk_t^{\star,g} + k_t^{\star,g} Z_t^g \cdot (1, 1-\theta_t^g)^\top \right) dt \\ - \rho_g dD_t^g - Z_t^g \cdot d\widetilde{M}_t^g, \tag{2}$$

$$dU_t^{b,c}(\theta^g, D^g) = \left( rU_t^{b,c}(\theta^g, D^g) - Bk_t^{\star,b,c} + k_t^{\star,b,c} Z_t^{b,c} \cdot (1, 1-\theta_t^g)^\top \right) dt \\ - \rho_g dD_t^g - Z_t^{b,c} \cdot d\widetilde{M}_t^g. \tag{3}$$

The **investor** designs two contracts:

$$\Psi_g = (D^g, \theta^g), \; \Psi_b = (D^b, \theta^b).$$

Each one of these contracts will depend on two state variables, due to the **temptation processes** of each agent.

**Contract designed for the good agent:**

$$dU_t^g(\theta^g, D^g) = \left( rU_t^g(\theta^g, D^g) - Bk_t^{\star,g} + k_t^{\star,g} Z_t^g \cdot (1, 1-\theta_t^g)^\top \right) dt \\ - \rho_g dD_t^g - Z_t^g \cdot d\widetilde{M}_t^g, \tag{2}$$

$$dU_t^{b,c}(\theta^g, D^g) = \left( rU_t^{b,c}(\theta^g, D^g) - Bk_t^{\star,b,c} + k_t^{\star,b,c} Z_t^{b,c} \cdot (1, 1-\theta_t^g)^\top \right) dt \\ - \rho_g dD_t^g - Z_t^{b,c} \cdot d\widetilde{M}_t^g. \tag{3}$$

- Value function of the **investor** on the lower boundary:
  $\Longrightarrow$ Obtained from the properties of the lower boundary.

- Value function of the **investor** on the upper boundary:
  $\Longrightarrow$ associated **system** of HJB equations (**ODE's**).

- Value function of the **investor** on the interior of the credible set:
  $\Longrightarrow$ associated **system** of HJB equations (**PDE's**).

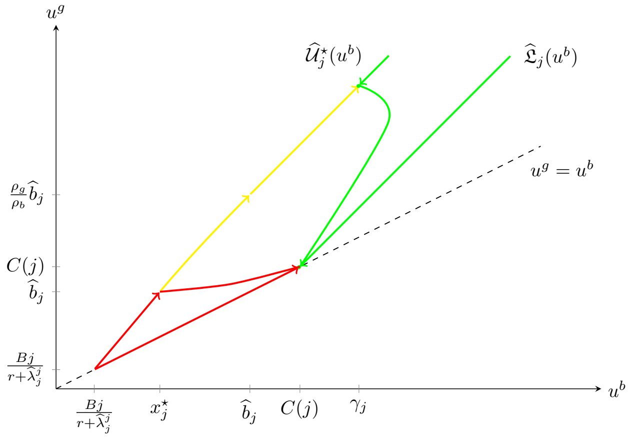



Figure : Optimal contract for the good agent.

- the bank is paid and the project is maintained.
- the bank is not paid and the project is liquidated.
- intermediate situations.

The value of the investor is given by

$$v_0 = \sup_{\{R_0 \leq u^b, u^{b,c} \leq u^b, u^{g,c} \leq u^g\}} p_g \widehat{V}_I^g(u^{b,c}, u^g) + p_b \widehat{V}_I^b(u^b, u^{g,c}).$$

Thank you for your attention!

Спасибо!

# Investor's problem and full-monitoring contract

PM: define some family of concave functions, unique solutions to the following system of ODEs

$$\begin{cases} \left(ru+\widehat{\lambda}_j^0\widehat{b}_j\right)(v_j^i)'(u)+j\mu-\widehat{\lambda}_j^0\left(v_j^i(u)-\dfrac{u-\widehat{b}_j}{\widehat{b}_{j-1}}v_{j-1}^i(\widehat{b}_{j-1})\right)=0,\ \ u\in\left(\widehat{b}_j,\widehat{b}_j+\widehat{b}_{j-1}\right], \\ \left(ru+\widehat{\lambda}_j^0\widehat{b}_j\right)(v_j^i)'(u)+j\mu-\widehat{\lambda}_j^0\left(v_j^i(u)-v_{j-1}^i(u-\widehat{b}_j)\right)=0, u\in\left(\widehat{b}_j+\widehat{b}_{j-1},\gamma_j^i\right], \\ \rho_i(v_j^i)'(u)+1=0, u>\gamma_j^i, \end{cases} \tag{4}$$

with initial values $\gamma_1^i:=\widehat{b}_1$ and

$$v_1^i(u):=\overline{v}_1^i-\frac{1}{\rho_i}(u-\widehat{b}_1), u\geq\widehat{b}_1,\ \ \overline{v}_1^i:=\frac{\mu}{\widehat{\lambda}_1^0}-\frac{\widehat{b}_1(r+\widehat{\lambda}_1^0)}{\rho_i\widehat{\lambda}_1^0},$$

and where for $j\geq 2$, $\gamma_j^i$ is defined recursively by $r/\widehat{\lambda}_j^0-1\in\partial v_{j-1}^i(\gamma_j^i-\widehat{b}_j)$, where $\partial v_{j-1}^i$ is the super–differential of the concave function $v_{j-1}^i$.

# Value function of the investor in the credible set

AS: We define, for any $t \geq 0$ and any $(u^{b,c}, u^g) \in \widehat{\mathcal{C}}_{I-N_t}$, the value function of the investor in the credible set by

$$V_t^g(u^{b,c}, u^g) := \operatorname*{ess\,sup}_{\Psi_{\boldsymbol{g}} \in \widehat{\mathcal{A}}^{\boldsymbol{g}}(t, u^{\boldsymbol{g}}, u^{\boldsymbol{b},\boldsymbol{c}})} \mathbb{E}^{\mathbb{P}^{\boldsymbol{k}^\star, \boldsymbol{g}(\Psi_{\boldsymbol{g}})}} \left[ \int_t^\tau (\mu(I - N_s)ds - dD_s^g) \middle| \mathcal{G}_t \right]. \quad (5)$$

The system of HJB equations associated to this control problem is given by $\widehat{V}_0^g \equiv 0$, and for any $1 \leq j \leq I$

$$\min\left\{-\sup_{\overline{C}^j}\left\{\begin{array}{l}\partial_{u^{b,c}}\widehat{V}_j^g(u^{b,c},u^g)\left(ru^{b,c}-Bk^{b,c}+[h^{1,b,c}+(1-\theta)h^{2,b,c}]\widehat{\lambda}_j^{k^{b,c}}\right)\\ +\partial_{u^g}\widehat{V}_j^g(u^{b,c},u^g)\left(ru^g-Bk^g+[h^{1,g}+(1-\theta)h^{2,g}]\widehat{\lambda}_j^{k^g}\right)\\ +[\widehat{V}_{j-1}^g(u^{b,c}-h^{1,b,c},u^g-h^{1,g})-\widehat{V}_j^g(u^{b,c},u^g)]\widehat{\lambda}_j^{k^g}\\ -\widehat{V}_{j-1}^g(u^{b,c}-h^{1,b,c},u^g-h^{1,g})(1-\theta)\widehat{\lambda}_j^{k^g}+\mu j\end{array}\right\}\right.$$

$$\left., \rho_b\partial_{u^{b,c}}\widehat{V}_j^g(u^{b,c},u^g)+\rho_g\partial_{u^g}\widehat{V}_j^g(u^{b,c},u^g)+1\right\}=0. \quad (6)$$

With $k^{b,c}=j\cdot 1_{\{h^{1,b,c}+(1-\theta)h^{2,b,c}<\widehat{b}_j\}}$, $k^g=j\cdot 1_{\{h^{1,g}+(1-\theta)h^{2,g}<\widehat{b}_j\}}$ and the set of constraints

$$\begin{aligned}\overline{C}^j \;=\; & \Big\{(\theta,h^{1,b,c},h^{2,b,c},h^{1,g},h^{2,g}),\theta\in[0,1],u^g=h^{1,g}+h^{2,g},\\ & u^{b,c}=h^{1,b,c}+h^{2,b,c},h^{2,g};h^{2,b,c}\geq\frac{B(j-1)}{r+\widehat{\lambda}_{j-1}^{SH}}\Big\}.\end{aligned}$$