System identification: response of dynamical systems in time frequency domain

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Outline

1. Generalities
   - Motivations
   - Signal representations
2. Joint representation of a signal (phase-space)
   - Wigner Distribution
   - Properties
3. Applications
   - Differential equations
   - Wave equation
4. Moments
   - Spatial moments of a pulse
   - Local moments
5. Future

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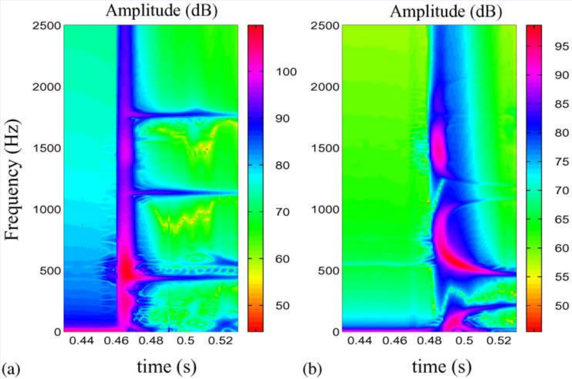
Motivations

1. Physical motivations
   - Response of systems to known inputs;
   - Propagation media: geometrical and physical parameters;
2. Mathematical motivations
   - Signals representations;
   - Dynamic in time frequency plane


Amplitude (dB)
Frequency (Hz)
2500
2000
1500
1000
500
0
0.44 0.46 0.48 0.5 0.52
time (s)
100
90
80
70
60
50
(a)
Amplitude (dB)
95
90
85
80
75
70
65
60
55
50
(b)

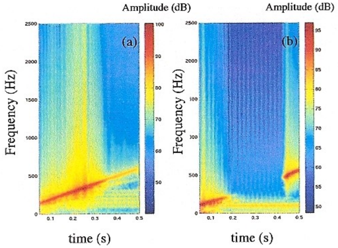


Figure 5: (a) Wigner-Ville transform of a FM signal upstream of the lattice (linear case). (b) Wigner-Ville transform of a FM signal downstream of the lattice (linear case).

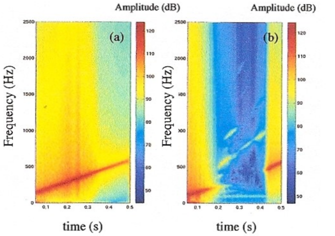


Figure 6: (a) Wigner-Ville transform of a FM sign upstream of the lattice (nonlinear case). (b) Wigne Ville transform of a FM signal downstream of th lattice (nonlinear case).

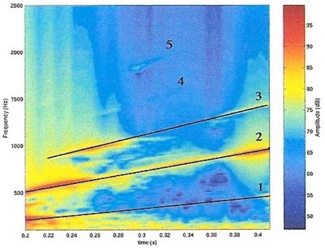


Figure 7: Zoom of the figure (6b) in the time range [0.2 : 0.42] s. Numbers label the corresponding harmonics from the fundamental (1) to harmonic 5.

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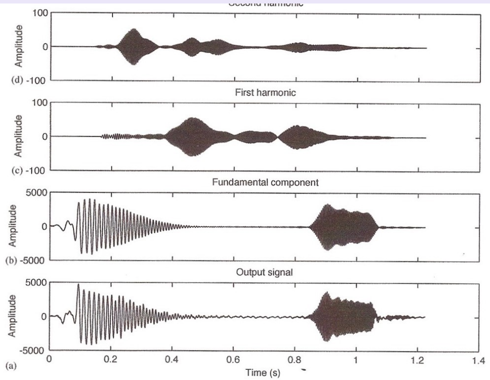

Fig. 8. Time waveforms of (a) transmitted signal downstream of the lattice, (b) fundamental component filtered in domain $\alpha = -$ (c) first harmonic filtered in domain $\alpha = -0.82$, (d) second harmonic filtered in domain $\alpha = -0.74$.

Signal representations

1. Time, time-space representations: $x(t)$, $u(x,t)$
2. Frequency representation : $X(f)$, $U(x,f)$

Question

How to construct a joint representation?

Response

Wigner did it in QM

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Wigner distribution

Definition (1 variable functions $x(t)$,y(t))

$$W_{xy}(t,\omega) = \frac{1}{2\pi}\int x^*(t-\tau/2)y(t+\tau/2)e^{-i\tau\omega}d\tau$$
$$W_{xy}(t,\omega) = \frac{1}{2\pi}\int X^*(\omega+\theta/2)Y(\omega-\theta/2)e^{-i\theta t}d\theta$$

Definition (2 variable functions $u(x,t)$,v(x,t))

$$W_{uv}(x,k,t) = \frac{1}{2\pi}\int u^*(x-\tau/2,t)v(x+\tau/2,t)e^{-ik\tau}d\tau$$
$$W_{uv}(x,k,t) = \frac{1}{2\pi}\int U^*(k+\theta/2,t)V(k-\theta/2,t)e^{-i\theta x}d\theta$$

More generally

$$W_{uv}(x,k,t,\omega) = \frac{1}{(2\pi)^2}\int\int u^*(x-\tau_x/2,t-\tau_t/2)v(x+\tau_x/2,t+\tau_t/2)e^{-ik\tau_x-i\omega\tau_t}d\tau_x d\tau_t$$

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More generally

$$W_{uv}(x,k,t,\omega) =$$
$$\frac{1}{(2\pi)^2} \int \int u^*(x-\tau_x/2, t-\tau_t/2) v(x+\tau_x/2, t+\tau_t/2) e^{-ik\tau_x - i\omega\tau_t} d\tau_x d\tau_t$$

Properties

Marginals

$\int W_{xx}(t,\omega)d\omega = |x(t)|^2, \int W_{xx}(t,\omega)dt = |X(\omega)|^2$

Energy

$\int\int W_{xx}(t,\omega)dtd\omega = \int |x(t)|^2 dt = \int |X(\omega)|^2 d\omega$

Recovery property

$x^*(t-\tau/2)y(t+\tau/2) = \int W_{xx}(t,\omega)e^{i\tau\omega}d\omega$

Derivatives

$W_{\dot{x}y} = \left(\frac{1}{2}\frac{\partial}{\partial t} - i\omega\right)W_{xy} = AW_{xy}, W_{x\dot{y}} = \left(\frac{1}{2}\frac{\partial}{\partial t} + i\omega\right)W_{xy} = BW_{xy}$

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Differential equations 1

Let the following equation:

$$L[x(t)] = f(t)$$

where $L[]$ is a linear differential operator
$f(t)$ is the driving term.
We seek the equation for Wigner distribution $W(xx)$

$$W_{L[x]L[x]}(t, w) = W_{ff}(t, \omega)$$

A particular case is

$$L[] = P(D) = \sum_{n=O}^{N} a_n D^n \quad \text{where} \quad D = \frac{d}{dt}$$

then

$$P^*(A)P(B)W_{xx} = W_{ff}.$$

Differential equations 2

If $g_{\alpha,\beta}(t,\omega)$ are eigenfunctions of operators $-iA$ and $-iB$ whith eigenvalues $i\alpha$ and $i(\alpha+2\beta)$ then

$$W_{xx}(t,\omega)=\int\int w_x(\alpha,\beta)g_{\alpha,\beta}(t,\omega)d<\alpha d\beta$$

and

$$W_{xx}(t,\omega)=\int\int \frac{w_f(\alpha,\beta)g_{\alpha,\beta}(t,\omega}{P^*(i\alpha)P(i(\alpha+2\beta))}d\alpha d\beta.$$

Differential equations 3

Exemple : Harmonic oscillator

$$\left(\frac{d^2}{dt^2} + 2\alpha\frac{d}{dt} + \omega_0^2\right)x(t) = f(t)$$

W_{xx} obeys the equation :

$$(A^2 + 2\alpha A + \omega_0^2)(B^2 + 2\alpha B + \omega_0^2)W_{xx} = W_{ff}$$

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Wave equation

General form

$$\sum_{n=0}^{N} a_n \frac{\partial^n u}{\partial t^n} - \sum_{n=0}^{M} b_n \frac{\partial^n u}{\partial x^n} = 0$$

An other form is the Dispersion Relation

$$\sum_{n=0}^{N} a_n (-i\omega)^n - \sum_{n=0}^{M} b_n (ik)^n = 0$$

Solutions

$$\omega = \Omega(k) \quad \text{or} \quad k = K(\omega)$$

Each solution is a **mode**

General solution

Let $u(x,0)$ be the IC. The solution is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int U(k,0) e^{ikx - i\Omega(k)t} dk$$

where

$$U(k,0) = \frac{1}{\sqrt{2\pi}} \int u(x,0) e^{-ikx} dx$$

If one defines $U(k,t) = U(k,0)e^{-i\Omega(k)t}$, then

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int U(k,t) e^{ikx} dx \\ U(k,t) &= \frac{1}{\sqrt{2\pi}} \int u(x,t) e^{-ikx} dk \end{aligned}$$

Consequences for Wigner distribution

$$W_{uu}(x,t) = \frac{1}{2\pi}\int U^*(k+\theta/2,t)U(k-\theta/2,t)e^{-i\theta x}d\theta$$

becomes

$$W_{uu}(x,k,t) = \frac{1}{2\pi}\int W(x',k,0)L(x-x',k,t)dx'$$

where

$$L(x-x',k,t) = \int e^{i(x-x')}e^{i\left(\Omega(k+\theta/2)-\Omega(k-\theta/2)\right)t}d\theta$$

keeping only the first order in θ, one obtains :

$$L(x-x',k,t) = \delta(x'-x+v(k)t)$$

and

$$W_{uu}(x,k,t) \approx W_{uu}(x-v(k)t,k,0)$$

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Moments of a pulse

Definition 1

$$< x^n >_t = \int x^n |u(x,t)|^2 dx$$

Definition 2

$$< x^n >_t = \int U^*(k,t)\mathcal{X}^n U(k,t) dk$$

where $\mathcal{X}$ is the position operator in the k representation :

$$\mathcal{X} = i\frac{\partial}{\partial k}$$

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Moments of a pulse

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Some results

Mean

$$\begin{aligned} < x >_t &= \int x|u(x,t)|^2 dt = \int U^*(k,t)\mathcal{X}U(k,t)dk \\ &= < x >_0 + Vt \quad \text{where} \quad V = \int \frac{d\Omega}{dk}|U(k,0)|^2 dk \end{aligned}$$

Second moment

$$< x^2 >_t = < x^2 >_0 + t < v\mathcal{X} + \mathcal{X}v >_0 + t^2 < v^2 >$$

Spread

$$\sigma^2_{x|t} = \sigma^2_{x|0} + 2tCov_{xv|0} + t^2\sigma^2_v$$

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Approach by Wigner (local moments)

In dispersive propagation

$$W_u(t,\omega,x) \approx W_u(t+K'(\omega),\omega,0),$$

and as Wigner distribution satisfies the marginals

$$< t_u^n >_x = \int t^n |u(x,t)|^2 dt = \int\int t^n W_u(t,\omega,x) d\omega dt$$

this suggests to define local central moments (centred about the frequency time shift)

Local duration

$$\begin{aligned}\sigma^2_{t|x,\omega} &= <(t- <t>_{x,\omega})^2>_{x,\omega} \\ &= \left(\int W_u(t,\omega,x)dt\right)^{-1}\int (t- <t>_{x,\omega})^2 W_u(t,\omega,x)dt\end{aligned}$$

where the local mean time $<t>_{x,\omega}$ is

$$<t>_{x,\omega} = \left(\int W_u(t,\omega,x)dt\right)^{-1}\int tW_u(t,\omega,x)dt$$

If

$$U(t,\omega) = B(t,\omega)e^{i\psi(x,\omega} = B(0,\omega)e^{i\psi(0,\omega)+K(\omega)x}$$

$$\begin{aligned}<t>_{x,\omega} &= -\psi'(0,\omega) - K'(\omega)x \\ \sigma^2_{t|x,\omega} &= \frac{1}{2}\Big(B'^2(0,\omega) - B''(0,\omega)B(0,\omega)\Big)/B^2(0,\omega) = \sigma^2_{t|0,\omega}.\end{aligned}$$

Higher order Local central Moments

One exemple

$$
\begin{aligned}
< (t- < t >_{x,\omega})^n >_{x,\omega} &= \left(\int W_u(t,\omega,x)dt\right)^{-1} \int (t- < t >_{x,\omega})^n W_u \\
&= \left(\int W_u(t,\omega,x)dt\right)^{-1} \int (t- < t >_{x,\omega})^n W_u \\
&= < (t- < t >_{0,\omega})^n >_{0,\omega}
\end{aligned}
$$

This moment is **approximatively invariant** to dispersion

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propagation in media with attenuation

what about others joint representations?