Probability Characteristics of the Absolute Maximum of the Gaussian Random Processes

By

Prof. Oleg V. Chernoyarov[1] and
Ass. prof. Alexandra V. Salnikova[2]

[1]National Research University "Moscow Power Engineering Institute",
Moscow, Russia
[2]Voronezh State Technical University, Voronezh, Russia

Introduction

Determination of the Gaussian random processes limiting characteristics is an important task in the fields of statistical radio physics and radio engineering, reliability theory, analysis of the extreme deviations and stability of technical systems, etc. In a number of studies, it is shown that the form of the distribution function

$$F(h) = P[\sup_{t \in [0,T]} \xi(t) < h]$$

of the absolute maximum of the Gaussian random process $\xi(t)$ depends on its continuous derivative. Below, the general formulas for distribution functions of the absolute maximum of the non-stationary differentiable and nondifferentiable Gaussian random process are presented.

Distribution of the absolute maximum of the differentiable Gaussian random process

Let us consider the non-stationary Gaussian random process $\xi(t)$

$$\xi(t): \quad m(t) = \langle \xi(t) \rangle \qquad B(t_1, t_2) = \langle [\xi(t_1) - m(t_1)][\xi(t_2) - m(t_2)] \rangle$$

and

$\xi(t), \ \dot{\xi}(t)$ are rms-continuous

We designate $\Pi(h,t)$ as the average numbers of the outliers of the realization $\xi(t)$ beyond the *h* level and within the elementary interval $[t, t+dt]$

We then presuppose that the threshold *h* is great enough, i.e.

$$h - m(t) >> \sigma(t) \qquad t \in [0, T] \qquad \sigma^2(t) = B(t,t)$$

In that case, the outliers flow of the realization $\xi(t)$ beyond the *h* level is reduced to the Poisson one. And the outliers for various elementary intervals will be approximately statistically independent.

Therefore, the probability of the *h* threshold noncrossing is equal to

$$P[\sup_{t\in[0,T]} \xi(t) < h] \approx \exp\left[-\tilde{\Pi}(h)\right], \qquad \tilde{\Pi}(h) = \int_0^T \Pi(h,t)\,dt$$

The general formula for the average number of outliers of the non-stationary Gaussian random process is known:

$$\Pi(h,t) = \left[\sqrt{B_2(t)}/2\pi\sigma^2(t)\right]\exp\left\{-[h-m(t)]^2/2\sigma^2(t)\right\}\times$$
$$\times\left\{\exp\left[-M_1^2(t)/2\right] + \sqrt{2\pi}M_1(t)\Phi[M_1(t)]\right\}.$$

Here

$$B_2(t) = \sigma^2(t)\left[\partial^2 B(t_1,t_2)/\partial t_1\partial t_2\right]_t - \left[\partial B(t_1,t_2)/\partial t_2\right]_t^2$$

$$M_1(t) = \frac{1}{\sigma(t)\sqrt{B_2(t)}}\left\{\sigma^2(t)\frac{dm(t)}{dt} + [h-m(t)]\left[\frac{\partial B(t_1,t_2)}{\partial t_2}\right]_t\right\}$$

$$\Phi(x) = \int_{-\infty}^{x}\exp\left(-u^2/2\right)du/\sqrt{2\pi}$$

In general, the function $P[\sup_{t\in[0,T]} \xi(t) < h]$ is not a nondecreasing function of *h*

Therefore, for an arbitrary *h* the following expression can be used as an approximation for the distribution function of the absolute maximum of the process $\xi(t)$

$$F(h) \approx \begin{cases} \exp\left[-\tilde{\Pi}(h)\right], & h \geq h_{\min}, \\ 0, & h < h_{\min}. \end{cases}$$

Here $h_{\min}$ is the least value of *h*, for which the inequality $\tilde{\Pi}(h) > \tilde{\Pi}(h+\varepsilon)$ is satisfied under any $\varepsilon > 0$

If values *h* and *T* are small, then this approximation can be rather crude. Since under $T \to 0$ the distribution of the largest values of the process $\xi(t)$ converges to the Gaussian distribution and we can use the approximation of the type

$$F(h) \approx \begin{cases} F_G(h)\exp\left[-\tilde{\Pi}(h)\right], & h \ge h_{\min}, \\ F_G(h_{\min})\exp\left[-\tilde{\Pi}(h_{\min})\right], & h < h_{\min}, \end{cases}$$

where $F_G(h) = \Phi\left[\left(h - m(0)\right)/\sigma(0)\right]$

If Gaussian random process is stationary one, i.e.

$$m(t) = m, \qquad \sigma^2(t) = \sigma^2, \qquad B(t_1, t_2) = B(t_2 - t_1)$$

then we have

$$F(h) \approx \begin{cases} F_G(h)\exp\left[-\dfrac{\alpha}{2\pi}\exp\left(-\dfrac{(h-m)^2}{2\sigma^2}\right)\right], & h \ge m, \\ F_G(m)\exp\left(-\dfrac{\alpha}{2\pi}\right), & h < m. \end{cases}$$

Here $\alpha = T/\tau_c$, $\tau_c = \left[\sigma\Big/\sqrt{-d^2B(\tau)/d\tau^2}\right]_{\tau=0}$

Distribution of the absolute maximum of the nondifferentiable Gaussian random process

Let us consider nondifferentiable Gaussian random process $\xi(t)$ with initial probability density

$$w(x;0) = \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{(x-m_0)^2}{2\sigma_0^2}\right], \qquad m_0 = m(0), \quad \sigma_0^2 = \sigma^2(0)$$

We are now focused on special but important case, when the process $\xi(t)$ is Markov random process with the constant drift and diffusion coefficients

$$F(h) = P\left[\sup \xi(t) < h\right] \Rightarrow F(h) = P\left[\eta(t) > 0\right], \qquad t \in [0,T]$$

where $\eta(t) = h - \xi(t)$

Then we can write down

$$F(h) = P[\eta(t) > 0] = \int_0^\infty w_\eta(z;T)\,dz$$

Here $w_\eta(z;t)$ is the one-dimensional probability density of the random process $\eta(t)$ realizations which have never reached the borders $z = 0$ and $z = \infty$ within the interval $[0,t]$

Due to the Markovian nature of the process $\eta(t)$, the function $w_\eta(z;t)$ can be found from the solution of the direct Fokker-Planck-Kolmogorov equation:

$$\frac{\partial w_\eta(z;t)}{\partial t} = -\frac{\partial}{\partial z}\left[K_{1\eta} w_\eta(z;t)\right] + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left[K_{2\eta} w_\eta(z;t)\right]$$

under the starting condition and the boundary conditions

$$w_\eta(z;0) = \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{(z - h + m_0)^2}{2\sigma_0^2}\right], \qquad w_\eta(0;t) = w_\eta(\infty;t) = 0$$

As a result, we have

$$w_\eta(z;t) = \frac{1}{\sqrt{2\pi K_{2\eta} t}} \int_0^\infty w_\eta(z_1;0) \exp\left[-\frac{K_{1\eta}^2}{2K_{2\eta}} t + \frac{K_{1\eta}}{K_{2\eta}} (z - z_1) \right] \times$$

$$\times \left\{ \exp\left[-\frac{(z - z_1)^2}{2K_{2\eta} t} \right] - \exp\left[-\frac{(z + z_1)^2}{2K_{2\eta} t} \right] \right\} dz_1$$

And the distribution function for the absolute maximum of Gaussian Markov process $\xi(t)$ with the constant drift K_1 and diffusion K_2 coefficients takes the form of

$$F(h) = \frac{1}{\sigma_0 \sqrt{2\pi}} \int_0^\infty \exp\left[-\frac{(z - h + m_0)^2}{2\sigma_0^2} \right] \times$$

$$\times \left[\Phi\left(\frac{z - K_1 T}{\sqrt{K_2 T}} \right) - \exp\left(\frac{2K_1 z}{K_2} \right) \left(1 - \Phi\left(\frac{z + K_1 T}{\sqrt{K_2 T}} \right) \right) \right] dz$$

When Gaussian Markov or local Markov random process $\xi(t)$ is the stationary one, then the required distribution function can be found in the following way. As shown by R.L. Stratonovich

$$P[\sup_{t\in[0,T]} \xi(t) < h] \approx \exp(-\rho T)$$

Where $\frac{1}{\rho} = \frac{2}{K_2}\int_{x_0}^{h}\frac{dx}{w_{st}(x)}$ and $w_{st}(x)$ is the stationary probability density of the random process $\xi(t)$

This formula was obtained for the case

$$K_2 T / 2\sigma_0^2 >> 1 \quad \text{and} \quad w_{st}(h) << 1 \quad (h >> m_0)$$

The value of x_0 is chosen in the region of maximum probability of the $\xi(t)$ process values, so that we can assume that $x_0 = m_0$

Using the asymptotic Laplace formula, we obtain the following expression for $h \to \infty$:

$$\frac{1}{\rho} = \frac{2\sigma_0^3}{K_2}\frac{\sqrt{2\pi}}{h - m_0}\exp\left[(h - m_0)^2 / 2\sigma_0^2\right]\left[1 + \mathrm{O}\left(h^{-1}\right)\right]$$

where $\mathrm{O}\left(h^{-1}\right)$ denotes terms of the order $1/h$

Therefore, for large *h*

$$P[\sup_{t \in [0,T]} \xi(t) < h] \approx \exp[-\delta\varphi(h)]$$

where $\varphi(h) = \frac{(h - m_0)}{\sigma_0\sqrt{2\pi}}\exp\left[-\frac{(h - m_0)^2}{2\sigma_0^2}\right], \quad \delta = \frac{K_2 T}{2\sigma_0^2}$

The accuracy of this approximation increases with δ and *h*.

Since $\exp[-\delta\varphi(h)]$ is a nondecreasing function only under $h \geq h_{\min}$, for the distribution function of the absolute maximum of the process $\xi(t)$ we use the approximation

$$F(h) = \begin{cases} \exp[-\delta\varphi(h)], & h \geq h_{\min}, \\ 0, & h < h_{\min}, \end{cases}$$

where $h_{\min}$ is the least value of *h* for which the inequality $\varphi(h) > \varphi(h+\varepsilon)$ is satisfied under any $\varepsilon > 0$. It is easy to see that $h_{\min} = m_0 + \sigma_0$

For not very large values of δ and *h*, the last expression can be somewhat refined by writing down

$$F(h) \approx \begin{cases} F_G(h)\exp[-\delta\varphi(h)], & h \geq h_{\min}, \\ F_G(h_{\min})\exp[-\delta\varphi(h_{\min})], & h < h_{\min}, \end{cases}$$

where $F_G(h) = \Phi[(h - m(0))/\sigma(0)]$. This approximation is asymptotically accurate for both $\delta \to \infty$ and $\delta \to 0$

CONCLUSION

The presented techniques allows finding out the distribution laws for the absolute maximum of the non-stationary Gaussian random processes, and these laws can be applied, with a corresponding generalization, for the determination of the limiting characteristics of the non-Gaussain random processes. The form of the distribution law for the absolute maximum depends on the fact whether the random process is differentiable, or nondifferentiable one.

Comparison with experimental data produced during the simulation in a number of particular cases shows us that the introduced formulas successfully describe the true distributions within a wide range of the random processes parameters values (for example, for differentiable stationary process – under arbitrary α and $h \geq m + \sigma$; for non-differentiable stationary process – under $\delta \geq 5$ and $h \geq h_{\min}$).

Thank you for your attention!