Hunt-Mackenhoupt-Wheeden Condition and Estimating of Pseudo-periodic Function

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Statistical problem

This talk is connected with nonparametric estimation of the function $s(t)$ as the observation process $y(t)$ is given by

$$dy(t) = s(t)dt + dx(t), t \in [-T, T].$$

Here unknown function s lies in a compact subset $\mathscr{L}_*$ of the Banach space $\mathscr{L}$ with the norm $\|\cdot\|_{\mathcal{L}}$,

$$\|s\|^2_{\mathcal{L}} = \sup_x \int\limits_x^{x+1} |s\,(t)\,|^2\, dt. \qquad (1)$$

The noise process $x(t)$ is the gaussian process with stationary increments with zero mean and with the spectral density f.

For a suitably chosen countable set Ψ of functions ψ, $\operatorname{supp}\psi \in [-T, T]$, we consider a discrete version of the statistical problem as we observe

$$[y, \psi] = [s, \psi] + [x, \psi]\,,\ \psi \in \Psi, \qquad (2)$$

where

$$[y,\psi] = \int\limits_{-\infty}^{\infty} \overline{\psi(t)}\, dy(t),\ [s,\psi] = \int\limits_{-\infty}^{\infty} \overline{\psi(t)}\, s(t)\, dt,\ [x,\psi] = \int\limits_{-\infty}^{\infty} \overline{\psi(t)}\, dx(t),$$

For an estimator $\widehat{s}_T$ of unknown function s we denote

$$R_T\left(\widehat{s}_T, f\right) = \sup_{s\in\mathscr{L}_*} \mathbf{E}_{s,f} \left\|\widehat{s}_T - s\right\|^2_{\mathscr{L}},$$

and by $\mathscr{R}_T\left(f\right)$ we denote the minimax risk,

$$\mathscr{R}_T\left(f\right) = \inf_{\widehat{s}_T} \sup_{s\in\mathscr{L}_*} \mathbf{E}_{s,f} \left\|\widehat{s}_T - s\right\|^2_{\mathscr{L}}.$$

One of the problems that we plan to discuss is how to choose in a reasonable way a system Ψ of functions ψ so as not to lose much in the rate of decrease in risk $R_T\left(\widehat{s}_T, f\right)$, as we assume that the spectral density f is unknown.

Stationary process and orthogonal projection operator

Let $x(t)$ be a gaussian process with zero mean $\mathbf{E}\, x(t) = 0$ and stationary increments. We use the notation

$$x[\varphi] = \int_{-\infty}^{\infty} \varphi(t)\, dx(t).$$

The linear operator $x[\varphi]$ is defined in the usual way on the indicator functions:

$$x\left[\mathbf{1}_{[a,b]}\right] = x(b) - x(a),$$

and well defined on linear span S of all such functions. The expected value of $|x\,[\varphi]|^2$ does not depend on the shift operator:

$$\mathbf{E}\, |x\,[\varphi(\cdot)]|^2 = \mathbf{E}\, |x\,[\varphi(\cdot + \tau)]|^2\,,$$

and therefore there exists a nonnegative measure μ such that

$$\mathbf{E}\, |x\,[\varphi(\cdot)]|^2 = \int_{-\infty}^{\infty} |\widehat{\varphi}|^2\, d\,\mu. \tag{3}$$

Here $\widehat{\varphi}(u)$ is the Fourier transform of φ,

$$\widehat{\varphi}(u) = \int_{-\infty}^{\infty} \varphi(t)e^{itu}\,dt,$$

and the spectral measure μ satisfies to the condition

$$\int_{-\infty}^{\infty} \frac{\mu(du)}{1+u^2} < \infty.$$

We assume that process x has the spectral density f. This means that

$$\mathbf{E}\,|x[\varphi]|^2 = \int_{-\infty}^{\infty} |\widehat{\varphi}(u)|^2\; f(u)\,du.$$

For a nonnegative function f which is defined on R, denote by L^2_f the Hilbert space with the inner product $(\cdot,\cdot)_f$ and the norm $\|\cdot\|_f$,

$$(h_1, h_2)_f = \int\limits_{-\infty}^{\infty} h_1(u)\overline{h_2(u)}\, f(u)\, du, \ \|h\|_f^2 = (h,h)_f\,.$$

In the case as $f \equiv 1$, we use the notation L^2 instead of $L^2_{\mathbf{1}}$, and $(\cdot,\cdot), \|\cdot\|$ instead of $(\cdot,\cdot)_{\mathbf{1}}, \|\cdot\|_{\mathbf{1}}$.

The linear operator $x[\varphi]$ defined on S can be extended to $\mathscr{D}_f$,

$$\mathscr{D}_f = \left\{\varphi :\ \varphi \in L^2_{loc},\ \widehat{\varphi} \in L^2_f\right\}, \tag{4}$$

where L^2_{loc} is the set of locally square summable functions. Denote $H(x)$ the subspace of the space $L^2(dP)$ generated by random variables $x[\varphi]$, $\varphi \in \mathscr{D}_f$. The relation

$$\pi\, x[\varphi] = \widehat{\varphi}$$

determines an isometry $\pi :\ H(x) \ \to\ L^2_f$. This allows to translate many of the problems of geometry in the space $H(x)$ into the appropriate analytical problems in the space L^2_f.

Orthoprojector on $H_T(x)$ in the space $L^2(dP)$

Denote $H_T(x)$ the subspace of the space $H(x)$ generated by random variables $x[\varphi]$, $\operatorname{supp}\varphi \in [-T,T]$. The subspace of the space L^2_f corresponding to the mapping π is denoted by $\mathscr{H}_T(f)$: $\mathscr{H}_T(f) = \pi\, H_T(x)$. In the case as

$$\int\limits_{-\infty}^{\infty} \frac{f(u)}{1+u^2}\, du < \infty$$

the subspace $\mathscr{H}_T(f) = \pi\, H_T(x)$ coincides with the closure of the linear span of the set

$$\left\{ \frac{1-e^{tu}}{u},\ |t| \le T \right\} \text{ since } \pi\, x\left[\mathbf{1}_{[a,b]}\right] = \frac{e^{bu}-e^{au}}{u}.$$

Denote $P_T(f)$ the orthoprojector (in the metric of the space $L^2(dP)$) onto $H_T(x)$. Let $\mathscr{P}_T(f)$ be the orthoprojector on $\mathscr{H}_T(x)$ in the space L^2_f. It is very difficult to construct the analytical representation of the operators $P_T(f)$ or $\mathscr{P}_T(f)$ for general spectral density f.

But in the case as $f \equiv 1$ the process $x(t)$ is a process with orthogonal increments. Therefore for any random variable ξ,

$$\xi = \int\limits_{-\infty}^{\infty} \varphi(t)\, dx(t), \text{ we have } P_{T}(f)\, \xi = \int\limits_{-T}^{T} \varphi(t)\, dx(t) = x\left[\mathbf{1}_{[T,T]}\varphi\right].$$

In other words in the case as $f \equiv 1$ for the process $x\,[\varphi]\,,\ \varphi \in \mathscr{D}_f$,

$$P_{T}(f)\, x\,[\varphi] = P_{T}\, x\,[\varphi] := x\left[\mathbf{1}_{[T,T]}\varphi\right] = \left(\mathbf{1}_{[T,T]}x\right)[\varphi]\,. \tag{5}$$

It is easy to see that in this case

$$\mathscr{P}_{T}(f)\, h\,(v) = \mathscr{P}_{T}\, h\,(v) := \int\limits_{-\infty}^{\infty} \frac{\sin T(v-u)}{\pi(v-u)}\, h(u)\, du. \tag{6}$$

The problem that will interest us in this section is the following: under what conditions on spectral density f the operator P_{T} gives a good approximation for the operator $P_{T}(f)$.

More precisely, we want to find conditions on the spectral density f for which

$$\mathbf{E}\,(x[\varphi] - P_{T}\,x[\varphi])^2 \leq C\,\mathbf{E}\,(x[\varphi] - P_{T}(f)\,x[\varphi])^2\,, \tag{7}$$

with constant $C = C(f)$ which depends only on f.

Clearly, that

$$\mathbf{E}\,(x[\varphi] - P_{T}\,x[\varphi])^2 \geq \mathbf{E}\,(x[\varphi] - P_{T}(f)\,x[\varphi])^2\,.$$

Passing to the operator norm $\|\cdot\|_{L^2(dP)}$ in space $L^2(dP)$, we must find out when

$$\|I - P_{T}\|_{L^2(dP)} \leq C(f)\ \|I - P_{T}(f)\|_{L^2(dP)} = C(f). \tag{8}$$

Evidently, we must find out when

$$\|P_{T}\|_{L^2(dP)} \leq c(f), \tag{9}$$

with constant $c(f)$ which depends only on f.

Orthoprojector on $\mathscr{H}_T(f)$ in the space L_f^2

Now we want to move to the corresponding analytic problem in the space L_f^2.

We recall that

$$P_T x[\varphi] = x[\mathbf{1}_{[-T,T]}\varphi], \text{ and } \pi\, x[\varphi] = \widehat{\varphi}.$$

Hence,

$$\pi\, P_T\, \pi^{-1} = \mathscr{P}_T\, h\,(v) := \int_{-\infty}^{\infty} \frac{\sin T(v-u)}{\pi(v-u)}\, h(u)\, du. \tag{10}$$

Since the operator P_T (acting in space $L^2(dP)$) and the operator $\mathscr{P}_T$ (acting in space L_f^2) are unitarily equivalent, then the condition (9) (described in the previous section) are equivalent to the condition

$$\|\mathscr{P}_T\|_{L_f^2} \le c(f), \tag{11}$$

where $\|\cdot\|_{L_f^2}$ is the operator norm in the space L_f^2.

We use notation H^2_+ for the Hardy space of analytic functions in the upper half-plane. For our purposes, we can think that H^2 is the subspace of the space L^2 which consists of functions g which is representable in the form

$$g(u) = \int\limits_0^\infty e^{iut}\,\varphi(t)\,dt.$$

For $t > 0$ we denote by $H^2_+(t)$ the subspace of the space L^2 which consists of functions g which is representable in the form

$$g(u) = \int\limits_0^t e^{iut}\,\varphi(t)\,dt.$$

Let P^+ be the operator of orthogonal projection in L^2 onto H^2_+, $P^+(t)$ be the operator of orthogonal projection onto $H^2_+(t)$, $P^- = I - P^+$.

Recall that the operator $\mathscr{P}_T$,

$$\mathscr{P}_T\, h\,(v) = \int\limits_{-\infty}^{\infty} \frac{\sin T(v-u)}{\pi(v-u)}\, h(u)\, du,$$

is the operator of orthogonal projection (in the metric of the space L^2) onto subspace $\mathscr{H}_T$ which consists of functions g which is representable in the form

$$g(u) = \int\limits_{-T}^{T} e^{iut}\, \varphi(t)\, dt.$$

Let ϵ_t be the operator of multiplication by the function $\epsilon_t(u) = e^{itu}$. It is clear, that $P^+(2T) = \epsilon_T\, \mathscr{P}_T\, \epsilon_{-T}$. Since operators $P^+(2T)$ and $\mathscr{P}_T$ are unitarily equivalent in the space L^2_f, for any nonnegative function f, then the condition (11) is equivalent to the condition

$$\sup_{T>0} \left\| P^+(2T) \right\|_{L^2_f} < \infty. \tag{12}$$

From the last condition we obtain that

$$\left\|P^{+}\right\|_{L_f^2} < \infty, \tag{13}$$

and using the equality

$$P^{+}(2T) = \epsilon_{2T}\, P^{-}\, \epsilon_{-2T} = \epsilon_{2T}\,(I - P^{+})\,\epsilon_{-2T},$$

we deduce that (12) and (14) are equivalent.

Let $\mathbb{H}$ denote the Hilbert transform,

$$\mathbb{H}\, g(x) = \frac{1}{\pi}\, p.v. \int\limits_{-\infty}^{\infty} \frac{g(t)}{x-t}\, dt := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int\limits_{|x-t|>\varepsilon} \frac{g(t)}{x-t}\, dt.$$

The operator $\mathbb{H}$ admits the representation

$$\mathbb{H} = -iP^{+} + iP^{-} = iI + 2iP^{+}.$$

The question we are interested here is under what condition on f the operator $\mathbb{H}$ is bounded in the weighted space L^2_f.

The answer is well known. The famous Hunt-Mackenhoupt-Wheeden theorem (1973) says that the Mackenhoupt condition

$$\sup_I \frac{1}{|I|} \int\limits_I f(u)\, du \, \frac{1}{|I|} \int\limits_I \frac{1}{f(u)}\, du < \infty$$

is necessary and sufficient for the boundedness of $\mathbb{H}$ in weigthed space.

Since $\mathbb{H} = iI - 2iP^+$, then the Mackenhoupt condition is is necessary and sufficient for the boundedness of operator P^+ in weighted space, and therefore for the uniform on $T > 0$ boundedness of the operators $\mathscr{P}_T$ in weighted space L^2_f. Recall that the operator P_T (acting in space $L^2(dP)$) and the operator $\mathscr{P}_T$ (acting in space L^2_f) are unitarily equivalent. We obtain the following theorem.

Theorem. Let x be a process with stationary increments with spectral density f. Then

$$\mathbf{E}\,\left(x[\varphi] - P_{T}\,x[\varphi]\right)^2 \le C\,\mathbf{E}\,\left(x[\varphi] - P_{T}(f)\,x[\varphi]\right)^2, \tag{14}$$

with constant $C = C(f)$ which depends only on f if and only if the spectral density f satisfies to the Mackenhoupt condition.

Statistical data

Throughout the talk we consider nonparametric estimation of the function $s(t)$ as the observation process $y(t)$ is given by

$$dy(t) = s(t)dt + dx(t), t \in [-T, T].$$

Here unknown function $s \in \mathscr{L}_* \subset \mathscr{L}$, where $\mathscr{L}$ is the Banach space with the norm $\|\cdot\|_{\mathcal{L}}$,

$$\|s\|_{\mathcal{L}}^2 = \sup_x \int_x^{x+1} |s\,(t)\,|^2\,dt < \infty. \tag{15}$$

As the parametric set $\mathscr{L}_*$ we take $\mathscr{L}_* = \mathscr{L}(\Lambda;\beta)$, where $\mathscr{L}(\Lambda;\beta)$ is the subset of the Stepanov class $\mathscr{L}(\Lambda)$ of pseudoperiodic functions

$$s(t) = \sum_{u\in\Lambda} a(u)e^{iut}, \ \sum_{u\in\Lambda} |a(u)|^2 < \infty, \tag{16}$$

which is defined by the condition

$$\sum_{u\in\Lambda}(1+|u|)^{2\beta}|a(u)|^2 \le C, \tag{17}$$

Λ is a countable subset of real line such that

$$\kappa = \kappa(\Lambda) = \inf_{u,v\in\Lambda,\, u\neq v} |u-v| > 0. \tag{18}$$

That is we assume that different points of the spectral set Λ are uniformly separated. Under this condition in the subspace $\mathscr{L}(\Lambda)$ of the Banach $\mathscr{L}$ there are two Hilbert norm which are topologically equivalent to the original norm $\|\cdot\|_{\mathscr{L}}$. Namely, for function $s \in \mathscr{L}(\Lambda)$,

$$s(t) = \sum_{u\in\Lambda} a(u)e^{iut},$$

denote

$$\|s\|_*^2 = \sum_{u\in\Lambda} |a(u)|^2, \;\; \|s\|_T^2 = \frac{1}{2T}\int\limits_{-T}^{T} |s(t)|^2\,dt.$$

R. Paley, N. Wiener (1932) proved that, for $s \in \mathscr{L}(\Lambda)$, $\kappa(\Lambda) > 0$,

$$c_1 \|s\|_* \le \|s\|_{\mathscr{L}} \le C_1 \|s\|_* , \tag{19}$$

and, for $T > T_0(\kappa)$,

$$c_2 \|s\|_{{}_T} \le \|s\|_{\mathscr{L}} \le C_2 \|s\|_{{}_T} . \tag{20}$$

Here c_1, c_2, C_1, C_2 – positive constant depend only on $\kappa = \kappa(\Lambda)$.

For some technical reasons it is more convenient to investigate the accuracy of the estimation in the norm $\|\cdot\|_*$. For an estimator $\widehat{s}_T$ of unknown function s we denote

$$R_T^*\left(\widehat{s}_T; \beta; f\right) = \sup_{s \in \mathscr{L}_*} \mathbf{E}_{s,f} \left\|\widehat{s}_T - s\right\|_*^2 ,$$

and by $\mathscr{R}_T^*\left(\beta; f\right)$ we denote the minimax risk,

$$\mathscr{R}_T^*\left(\beta; f\right) = \inf_{\widehat{s}_T} \sup_{s \in \mathscr{L}_*} \mathbf{E}_{s,f} \left\|\widehat{s}_T - s\right\|_{\mathscr{L}}^2 .$$

It is clear that, under the condition $\kappa(\Lambda) > 0$,

$$c\, R_T^*\left(\widehat{s}_T; \beta; f\right) \le R_T\left(\widehat{s}_T; \beta; f\right) \le C\, R_T^*\left(\widehat{s}_T; \beta; f\right) , \tag{21}$$

and

$$c\,\mathscr{R}^*_T\,(\beta;f) \le \mathscr{R}_T\,(\beta;f) \le C\,\mathscr{R}^*_T\,(\beta;f)\,. \tag{22}$$

Here constants c, C depend only on κ. Recall, that $R_T\,(\widehat{s}_T;\beta;f)\,,\mathscr{R}_T\,(\beta;f)$ are risk and minimax risk, as we measure the accuracy of the estimation in the norm $\|\cdot\|_{\mathscr{L}}$.

Let L^2_T be the L^2-space on the interval $[-T,T]$ constructed by the normalized Lebesque measure, and $\|\cdot\|_{T},\ (\cdot,\cdot)_{T}$ are the norm and scalar product in L^2_T.

Below we use the notation

$$y[\psi] = \frac{1}{2T}\int_T^T \overline{\psi(t)}\,dy(t),\ s[\varphi] = \frac{1}{2T}\int_T^T \overline{\psi(t)}\,s(t)\,dt,\ x[\varphi] = \frac{1}{2T}\int_T^T \overline{\psi(t)}\,dx(t).$$

We want to choose in a optimal way a countable set Ψ and continue estimating in a discrete scheme, as we observe

$$y[\psi] = s[\psi] + x[\psi],\ \psi \in \Psi,$$

and try to construct an estimator $\widehat{s}$ for unknown $s \in \mathscr{L}$ on data $\{y[\psi],\ \psi \in \Psi\}$.

Denote $\varphi_u(t) = e^{iut}$. Introduce the countable set of function $\Phi = \{\varphi_u,\, u \in \Lambda\}$. The system Φ is the Riesz basis of the subspace $\mathscr{L}(\Lambda)$ (more precisely the restriction of a subspace $\mathscr{L}(\Lambda)$ on interval $[-T, T]$) in the metric of the space L^2_T). There is the conjugate system $\Psi_T = \{\psi_u,\, u \in \Lambda\}$, which is defined by

$$(\varphi_u, \psi_v)_T = \delta_{u,v}.$$

Therefore, for function $s(t) = \sum\limits_{u \in \Lambda} a(u) e^{iut}$,

$$(s, \psi_u)_T = a(u).$$

Therefore,

$$y\left[\psi_u\right] = a(u) + x\left[\psi_u\right],\ u \in \Lambda. \tag{23}$$

It should be noted that the functions $\psi_u = \psi_u^T$ depend on T and the system Ψ_T is uniquely determined only if we require that

$$\psi_u^T \in \mathbf{1}_{[-T,T]} \mathscr{L}(\Lambda),\ u \in \Lambda.$$

In this case, under condition $\kappa(\Lambda) > 0$, the following inequality holds

$$\|\psi_u^T\|_T \leq C(\kappa).$$

On observations

$$y\left[\psi_u\right] = a(u) + x\left[\psi_u\right], \; u \in \Lambda \tag{24}$$

we will construct an estimate $\widehat{\mathbf{a}}_T = (\widehat{a}_T(u), \, u \in \Lambda)$ of the coefficient vector $\mathbf{a} = (a(u), \, u \in \Lambda)$, and then will construct an estimate

$$\widehat{s}_T(t) = \sum_{u \in \Lambda} \widehat{a}_T(u) e^{iut},$$

using a priori information that

$$\sum_{u \in \Lambda} |a(u)|^2 \left(1 + |u|\right)^{2\beta} \leq C. \tag{25}$$

Since $\|s - \widehat{s}_T\|_*^2 = \|\widehat{\mathbf{a}}_T - \mathbf{a}\|_2^2 := \sum\limits_{u \in \Lambda} |\widehat{a}_T(u) - a(u)|^2$, we have almost the same statistical problem as the initial one.

However, with a small loss of information: we do not use observations $y[\psi]$, as $s[\psi] = 0$, for all $s \in \mathscr{L}(\Lambda)$.

We denote the minimax risk $\mathcal{R}_T(\beta; f)$ in the problem of estimating the vector **a** described above, and give the conditions under which the loss of information mentioned above will not be catastrophic.

Theorem. Suppose that $\kappa(\Lambda) > 0$, $\beta > 0$, and the spectral density f of process x satisfies to the condition

$$\lambda(f) := \sup_I \frac{1}{|I|} \int\limits_I f(u)\, du \, \frac{1}{|I|} \int\limits_I \frac{1}{f(u)}\, du < \infty. \tag{26}$$

Then

$$c\,\mathcal{R}_T(\beta; f) \le \mathscr{R}_T(\beta; f) \le C\,\mathcal{R}_T(\beta; f), \tag{27}$$

where positive constants c, C depend only on $\kappa(\Lambda), \lambda(f)$.

Discrete model

Let us consider in detail the problem of estimating an unknown vector $\mathbf{a} = (a(u),\ u \in \Lambda)$ from observations

$$Y_u = a(u) + X_u,\ u \in \Lambda,\ \mathbf{a} \in \Theta. \tag{28}$$

Here $\mathbf{X} = (X_u,\ u \in \Lambda)$ is a gaussian vector with zero mean and $\mathbf{E}X_u = \sigma_u^2$. Denote $\sigma = \left(\sigma_u^2,\ u \in \Lambda\right)$ and let $\mathscr{R}(\Theta; \sigma)$ be the mimimax risk.

In the case that interests us the set Θ is defined by

$$\sum_{u \in \Lambda} |a(u)|^2 \,(1 + |u|)^{2\beta} \le C; \tag{29}$$

$$X_u = x\left[\psi_u^T\right],\ \text{where the system } \Psi = \{\psi_u^T,\ u \in \Lambda\} \tag{30}$$

is defined by

$$\frac{1}{2T} \int_{-T}^{T} \psi_u^T(t)\, e^{-ivt}\, dt = \delta_{u,v},$$

and the Gaussian process x has zero mean and spectral density f. So, the random variables X_u are gaussian with zero mean and

$$\sigma_u^2 := \mathbf{E} X_u^2 = \frac{1}{4T^2} \int\limits_{-\infty}^{\infty} |\widehat{\psi}_u^T(z)|^2 \, dz.$$

The case, when $\{X_u,\ u \in \Lambda\}$ are independent Gaussian variables, and Θ is compact centrally symmetric subset of the space l^2 was well investigated (I. Ibragimov and R. Hasminskii (1984), D. Donoho and all (1990)). The possibility of transition to dependent variables is given by the following lemma which belong to S. Reshetov.

Lemma 1. Let $X = (X_u,\ u \in \Lambda)$ be a Gaussian vector with zero mean. Suppose that there exists a constant $c(X)$ such that for any finite set $\{a(v),\ v \in \Lambda\}$

$$\mathbf{E}\, |X_u - \sum_{v \neq u} a(v) X_v|^2 \geq c(X)\, \mathbf{E} |X_u|^2. \tag{31}$$

Then there exists a constant $C > 0$ which depend only on $c(X)$ that

$$C \sum_{u \in \Lambda} \tau_u^2 \wedge \sigma_u^2 \leq \mathscr{R}(\Theta; \sigma) \tag{32}$$

Choice of conjugate system

Note that if $\kappa(\Lambda) > 0$, then, for $T > T_0$, the operator of multiplication by the indicator function $\mathbf{1}_{[-T,T]}(t)$ Is a bounded and boundedly invertible operator from $\mathscr{L}(\Lambda)$ (considered as a subspace of a Banach space $\mathscr{L}$) into the subspace of space L^2_T, which defined by $\mathscr{L}_T(\Lambda) = \mathbf{1}_{[-T,T]}\,\mathscr{L}(\Lambda)$. In the future, it will be convenient for us to assume that all functions from L^2_T are equal to zero outside the interval $[-T, T]$.

We use the notation

$$\varphi_u(T;t) = \mathbf{1}_{[-T,T]}(t)\, e^{iut}.$$

Let $\{g_u^r,\ u \in \Lambda\}$ be the system, from the space $\mathscr{L}_r(\Lambda)$, which is conjugate to the system $\{\varphi_u(r;\cdot),\ u \in \Lambda\}$. That is

$$\int\limits_{-r}^{r} g_u^r(t)\, e^{-ivt}\, dt = 2r\delta_{u,v} \text{ if } v \in \Lambda.$$

For a fixed $r > T_0$ and $T > r$, we define the new system $\{\psi_u^T,\ u \in \Lambda\}$ by

$$\psi_u^T(t) = \frac{T}{2r(T-r)} \int\limits_{-\infty}^{\infty} g_u^r(t-s)\, \varphi_u(T-r;s)\, ds. \tag{33}$$

Lemma 2. Functions $\psi_u^T \in L_T^2$, and, for $v \in \Lambda$,

$$\frac{1}{2T} \int\limits_{-T}^{T} \psi_u^T(t)\, e^{-ivt}\, dt = \delta_{u,v}. \tag{34}$$

Class $\mathscr{K}$ of spectral densities

For $\alpha > -1$, $\beta > 1$ and $0 < b \le B$, we introduce the class $A(\alpha, \beta; b, B)$ of spectral densities by the conditions

$$b\,\varepsilon^{\alpha} \;\le\; \frac{\sum\limits_{u\in\Lambda,\,|u|\le m} f_\varepsilon(u)\,(1+|u|)^{2\beta}}{\sum\limits_{u\in\Lambda,\,|u|\le m} (1+|u|)^{2\beta}}\,,\;\; \frac{1}{N(m)} \sum_{u\in\Lambda,\,|u|\le m} f_\varepsilon(u) \le B\,\varepsilon^{\alpha}. \tag{35}$$

Here $N(m)$ is the number of points from Λ, contained in the interval $[-m, m]$.

Class of spectral densities $\mathscr{K} = \mathscr{K}(\alpha, \beta; b, B); \lambda)$ Is distinguished from the class $A(\alpha, \beta; b, B)$ be the condition

$$\lambda(f) \le \lambda < \infty. \tag{36}$$

Condition on spectral set Λ

We assume that

$$\kappa(\Lambda) = \inf_{u,v\in\Lambda,\, u\neq v} |u-v| > 0$$

and points of the spectral set Λ in the following sense regular distributed over large intervals $[-m, m]$: for positive c and $m > m_0$,

$$m^{2\beta+1} \le \sum_{u\in\Lambda,\, |u|\le m} (1+|u|)^{2\beta}. \tag{37}$$

The class of such spectral sets will be denoted by $\mathscr{B}(\beta;\kappa)$

Asymptotically optimal estimate

Let the estimator $\widehat{s}_T$ is defined by

$$\widehat{s}_T = \sum_{u\in\mathscr{L},|u|\le m(T)} \widehat{a}_T(u), \tag{38}$$

where $m(T) = T^{\frac{1+\alpha}{1+2\beta}}$, and $\widehat{a}_T(u)$ is unbiased estimator of the coefficient $a(u)$,

$$\widehat{a}_T(u) = y[\psi_u^T],$$

which is constructed on the specially selected the conjugate system $\Psi = \{\widehat{a}_T(u),\ u \in \Lambda\}$,

$$\psi_u^T(t) = \frac{T}{2r(T-r)} \int\limits_{-\infty}^{\infty} g_u^r(t-s)\,\varphi_u(T-r;s)\,ds.$$

Theorem. Suppose unknown function $s \in \mathscr{L}(\Lambda;\beta)$, x Gaussian process with zero mean and spectral density $f \in A(\alpha,\beta;b,B)$, spectral set Λ belongs to the class $\mathscr{B}(\beta;\kappa)$. Then, for positive constant c, C,

$$c\,R_T(\widehat{s}_T;\beta) \le \mathscr{R}_T(\beta) \le C\,T^{-\frac{(1+\alpha)(2\beta)}{1+2\beta}}. \tag{39}$$

Vector valued stationary process

Let f be a $d \times d$ matrix weight, that is a function on real line $\mathbb{R}$ whose values are selfadjoint nonnegative matrices. We define a weighted space $L^2(f)$ as the space of all measurable $\mathbb{C}^d$-valued functions on $\mathbb{R}$ satisfying to the condition

$$\|g\|_{L^2(f)} = \int_{-\infty}^{\infty} \big(f(u)g(u), g(u)\big)\, du < \infty. \tag{40}$$

We will use the notation $L^2(\mathbb{C}^d)$, as f is identity matrix. So, $L^2(\mathbb{C}^d)$ is the space of square summable functions on real line with values in $\mathbb{C}^d$. We denote $H^2(\mathbb{C}^d)$ the corresponding Hardy space of analytic functions. For our purposes, we can think that H^2 is the subspace of $L^2(\mathbb{C}^d)$ which consists of functions $g = (g_1, \ldots, g_d)$ with coordinate which is representable in the form

$$g(u) = \int_0^{\infty} e^{iut}\, \varphi(t)\, dt.$$

We shall use the same notation, as in the case $d = 1$. Let P_+ be the orthogonal projection in L^2 onto H^2, $P_- = I - P - +$.

Consider the Hilbert Transform,

$$\mathbb{H} = -iP_+ + iP_- = iI + 2iP_+.$$

S. Treil and A. Volberg proved (1995) that vector Muckenhoupt condition

$$\sup_I \left\| \left(\frac{1}{|I|} \int\limits_I f(u)\, du \right)^{1/2} \left(\frac{1}{|I|} \int\limits_I f^{-1}(u)\, du \right)^{1/2} \right\| < \infty. \tag{41}$$

is necessary and sufficient for the boundedness of Hilbert Transform $\mathbb{H}$ in $L^2(f)$ with matrix weight.

Estimation problem in vector valued case

Now consider the case, as we observe vector valued process $y(t) = (y_1(t), y_2(t))$ which is given by

$$dy_1(t) = s_1(t)dt + dx_1(t), t \in [-T, T],$$

$$dy_2(t) = s_2(t)dt + dx_2(t), t \in [-T, T],$$

Here unknown functions $s_j \in \mathscr{L}_*(j) \subset \mathscr{L}(\Lambda_j),\ j = 1, 2,$

$\mathscr{L}_*(j)$ is the subset of the Stepanov space $\mathscr{L}(\Lambda_j)$ of pseudoperiodic functions

$$s(t) = \sum_{u \in \Lambda} a(u)e^{iut}, \text{ defined by } \sum_{u \in \Lambda} (1 + |u|)^{2\beta_j} |a(u)|^2 \leq C, \tag{42}$$

The noise process $x(t) = (x_1(t), x_2(t))$ is the gaussian process with stationary increments with zero mean and with the spectral density f.

We consider the problem of estimating function s_1 with nuisance parameters s_2, and denote by $\mathscr{R}_T(f)$ the minimax risk of this estimation problem.

More precisely, we consider a simple but non-trivial case when

$$f(u) = \begin{pmatrix} 1 & p(u) \\ \overline{p(u)} & 1 \end{pmatrix}. \tag{43}$$

Theorem . Suppose $\Lambda_1 \cap \Lambda_2 = 0$, $\beta_1 = \beta_2 = \beta$, spectral density f satisfies to the vector Muckenhoupt condition. Then

$$\mathscr{R}_T(f;\beta) \leq C R_T(1-|p|^2;\beta).$$