On asymptotically minimax nonparametric estimation and detection of signal in Gaussian white noise. Maxiset approach

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Plan of the talk

We remind the definition of maxisets in nonparametric estimation. The assignment of maxisets of linear procedures will be provided (Kerkacharian and Picard (1992) and Rivoirard (2004)).

We introduce the notion of maxiset for the problems of nonparametric hypothesis testing.

The maxisets of the most widespread nonparametric test statistics will be established: sum of squares of estimators of Fourier coefficients, L_2-norms of kernel estimators, χ^2-tests, Kramer- von Mises tests.

The asymptotically minimax tests and estimators on maxisets are established (earlier for such functional sets the results were known only for wavelet bases).

Model

$$dY_n(t) = f(t)dt + \frac{\sigma}{\sqrt{n}}dw(t), \quad t \in (0,1), \sigma > 0 \qquad (1)$$

Maxisets. Nonparametric Estimation

Estimator $\hat{f}_n$ has minimax rate of convergence n^{-r} on the set V if

$$C_1 n^{-2r} \le \sup_{f \in V} E||\hat{f}_n - f||^2 \le C_2 n^{-2r} \qquad (2)$$

The set V is called n^{-r}-maxiset for estimator $\hat{f}_n$ if the following statement holds.

The estimator has minimax rate of convergence n^{-r} on the set U iff $U \subset \lambda V$ for some $\lambda > 0$.

Maxisets of Linear Procedures

Kerkacharian and Picard (1992) showed that balls $B^s_{2\infty}(P_0) = H(s, P_0)$ in Besov space $B^s_{2\infty}$ with $r = \frac{s}{1+2s}$ are n^{-r}-maxisets for kernel and projection estimators.

$$\int (f^{(l)}(x+t) - f^{(l)}(x))^2 dx \le L|t|^{2(s-l)}$$

where $l = [s]$.

For the proof Kerkacharian and Picard (1992) have implemented wavelet technique. For the wavelet basis the balls in Besov spaces $B^s_{2\infty}$ have the following assignment

$$B^s_{2\infty}(P_0) = \left\{ f : f = \sum_{k=1}^{\infty} \sum_{j=1}^{2^k} \theta_{kj}\phi_{kj}, \ \sup_{\lambda>0} 2^{2\lambda s} \sum_{k>\lambda}^{\infty} \sum_{j=1}^{2^k} \theta_{kj}^2 \le P_0 \right\}.$$

Rivoirard (2004) showed that for linear projection estimators satisfying some weak assumptions the maxisets are described in following form

$$H(s,P_0) = B^s_{2\infty}(P_0) = \left\{ f : f = \sum_{j=1}^{\infty} \theta_j \phi_j,\ \sup_{\lambda>0} \lambda^{2s} \sum_{j>\lambda} \theta_j^2 < P_0 \right\}.$$

He studied projection estimators building on arbitrary orthogonal systems of functions. For wide class of orthogonal systems of functions the sets $B^s_{2\infty}(P_0)$ are balls in Besov spaces $B^s_{2\infty}$.

Maxisets in nonparametric hypothesis testing

$$dY_n(t) = f(t)dt + \frac{\sigma}{\sqrt{n}} dw(t), \quad t \in (0,1), \sigma > 0 \tag{3}$$

The problem is to test the hypothesis

$$H_0 : f(x) = 0, \quad x \in (0,1)$$

versus nonparametric alternatives

$$H_n : f \in V_n = V_n(P_0) = \{f : ||f|| \geq cn^{-r}, f \in P_0 U\}$$

For any test $K_n = K_n(X_1, \dots, X_n)$ denote $\alpha(K_n)$ its type I error probability and $\beta(K_n, f)$ its type II error probability for the alternative $f \in L_2(0,1)$. Denote

$$\beta(K_n, V_n) = \sup\{\beta(K_n, f), f \in V_n\}.$$

We say that, for the test statistics $T_n(Y_n)$, the problem of signal detection is n^{-r}-distinguishable on the set $P_0 U$ if there is sequence of tests K_n generated $T_n(Y_n)$ such that

$$\limsup_{n\to\infty}(\alpha(K_n) + \beta(K_n, V_n)) < 1 \qquad (4)$$

Desirable definition of maxisets and maxispaces

We we want to find the functional Banach space $L \subset L_2(0,1)$ such that

problem of signal detection is n^{-r}-distinguishable on the ball in L

For any $f \notin L, f \in L_2(0,1)$, there are functions $f_{1n}, \ldots, f_{k_n n} \in L$ such that

$$||f - \sum_{i=1}^{k_n} f_{in}|| \geq cn^{-r}$$

and

$$\beta\left(K_n, f - \sum_{i=1}^{k_n} f_{in}\right) \to 1 - \alpha, \quad \alpha(K_n) = \alpha \tag{5}$$

(the right-hand side of (**??**) may be greater then $1 - \alpha$. We shall consider the worst case.)

L contains the functions of the largest possible smoothness for this setup.

Empirical Discussion

Let us discuss the content of the first two points of this definition.

Let $f \notin L$. Then there are functions $f_{1n}, \dots, f_{k_n n} \in L$ such that

$$||f - \sum_{i=1}^{k_n} f_{in}|| \geq cn^{-r}$$

and

$$\beta\left(K_n, f - \sum_{i=1}^{k_n} f_{in}\right) \to 1 - \alpha, \quad \alpha(K_n) = \alpha \tag{6}$$

However, if $f_{in} \in V_n(P_{in}), P_{in} \to \infty$ then $\beta(K_n, f_{in})$ may also tends to $1 - \alpha$. Thus, if we take $f = 0$ and implement such a definition, we get that $f = 0 \notin L$.

Conclusion

I see two ways of solution of this problem.

i. to prove that

$$\beta(K_n, f - \sum_{i=1}^{k_n} f_{in}) \to 1 - \alpha$$

faster then

$$\beta(K_n, f_{in}) \to 1 - \alpha$$

ii. Introduce some limitations on functions f_{in}

We shall consider more simple *ii.* We suppose that functions f_{in} should belong to specially defined finite dimensional spaces Π_i. These spaces are constructed by unit ball U of maxispace L.

Third point of definition

We can take arbitrary sequence of unsmooth functions and search for the maxispace L containing these functions. Thus the maxiset problem is ambiguously defined without the last condition.

Preliminary definition and notation

Let $L \subset L_2(0,1)$ be Banach space with norm $||\cdot||_L$ and let $U = \{x : ||x||_L \le 1, x \in L\}$, be the unit ball in L.

Denote $d_1 = \max\{||x||, x \in U\}$ and denote e_1 vector $e_1 \in U$ such that $||e_1|| = d_1$. Roughly speaking, vector e_i is vector of U on which i-width attains their value.
The further definition has inductive character. For $i = 2, 3, \dots$ denote $d_i = \max\{||x||, x \in U, < x, e_j >= 0, 1 \le j < i\}$. Define vector e_i such that $||e_i|| = d_i, e_i \in U, < e_i, e_k >= 0$ for $k = 1, \dots, i-1$.

Denote Π_i linear space generated vectors $e_1, \dots, e_i$. For any $x \in L_2(0,1)$ denote x_{Π_i} the projection of vector x on subspace Π_i and $\tilde{x}_i = x - x_{\Pi_i}$. Such a definition allows us to study the behaviour of "the tail" of the vector x.

Formal maxiset definition

We say that L is maxispace and $\mu U, \mu > 0$ is maxiset for test statistics T_n generating sequence of tests K_n, $\alpha(K_n) = \alpha(1 + o(1)), 0 < \alpha < 1$, if there holds

$$\limsup_{n\to\infty}(\alpha(K_n) + \beta(K_n, V_n(\mu)) < 1 \tag{7}$$

and for any $x \notin L, x \in L_2(0,1)$, there are sequences i_n, j_{i_n} such that $||\tilde{x}_{i_n}|| > c j_{i_n}^{-r}$ and

$$\limsup_{n\to\infty}(\alpha(K_{i_n}) + \beta(K_{i_n}, \tilde{x}_{i_n})) \geq 1 \tag{8}$$

Remarks

Suppose that functions $e_1, e_2, \dots$ are sufficiently smooth. Then, considering the functions $\tilde{x}_i = x - x_{\Pi_i}$ we "in some sense delete a smooth part of function x and study the behaviour of remaining oscillating part."

In definition of maxispace we associate with each $x \in L_2(0,1)$ vectors $\tilde{x}_i$ having small norms and cover by our consideration all space $L_2(0,1)$

In all further setups we show that the arising maxispaces are Besov spaces $B^s_{2\infty}$. For quadratic tests we have more general situation. The assignment of maxispaces in some orthonormal basis coincide with the assignment in trigonometric basis of Besov spaces $B^s_{2\infty}$.

Maxispaces for quadratic test statistics

We consider a problem of signal detection on a sequence space. We observe

$$y_j = \theta_j + \frac{\sigma}{\sqrt{n}}\xi_j, \quad 1 \le j < \infty \tag{9}$$

where $y_j = \int \phi_j dY_n(t), \theta_j =< f, \phi_j >, \xi_j = \int \phi_j dw(t)$.

The problem is to test the hypothesis $H_0 : f = 0$ versus alternative $H_n : f \in V_n$.

The test statistics are the quadratic forms

$$T_n(Y_n) = \sum_{j=1}^{\infty} \kappa_{jn} y_j^2 - \sigma^2 n^{-1} \sum_{j=1}^{\infty} \kappa_{jn}^2$$

with some coefficients $\kappa_{jn}^2 > 0$

Since the talk contains a lot of results Theorem will be provided only for the test statistics having the following form

$$T_n(Y_n) = \sum_{j=1}^{k_n} y_j^2 - \sigma^2 n^{-1} k_n$$

Theorem

Denote $s = \frac{r}{2-4r}$. Then $r = \frac{2s}{4s+1}$.

The space $B^s_{2\infty}$ is maxispace for the test statistics $T_n(Y_n)$ with $k_n \asymp n^{2-4r} = n^{\frac{2}{1+4s}}$.
Here

$$H(s,P_0) = B^s_{2\infty}(P_0) = \left\{ f : f = \sum_{j=1}^{\infty} \theta_j \phi_j,\ \sup_{\lambda>0} \lambda^{2s} \sum_{j>\lambda} \theta_j^2 < P_0 \right\}.$$

Kernel-based tests

We consider the problem of signal detection on a circle.
Define kernel estimator

$$\hat{f}_n(t) = \frac{1}{h_n} \int_{-\infty}^{\infty} K\left(\frac{t-u}{h_n}\right) dY_n, \quad t \in (0,1)$$

where h_n is a sequence of positive numbers, $h_n \to 0$ as $n \to 0$.
The kernel K is bounded function such that the support of K is contained in $[-1,1]$, $K(t) = K(-t), t \in R^1$ and $\int K(t)dt = 1$.
We consider the kernel based tests with test statistics

$$T_n(Y_n) = n^{-1} h_n^{1/2} \sigma^{-1} (||\hat{f}_n||^2 - (nh_n)^{-1} ||K||^2)$$

where

$$\sigma^2 = \int \left(\int K(t-s)K(s)ds \right)^2 dt.$$

Theorem

For the kernel-based tests with $h_n \asymp n^{4r-2} = n^{\frac{-2}{1+4s}}$ Besov spaces $B^s_{2\infty}$ with $s = \frac{r}{2-4r}$ are n^{-r}-maxispaces. Here $r = \frac{2s}{4s+1}$

Chi-squared tests

Let $X_1, \ldots, X_n$ be i.i.d.r.v.'s with c.d.f. $F(x), x \in (0,1)$. Let c.d.f. $F(x)$ has a density $f(x) = dF(x)/dx, x \in (0,1)$. Suppose that $f \in L_2(0,1)$ with the norm

$$||f|| = \left(\int_0^1 f^2(x) dx \right)^{1/2} < \infty.$$

We explore the problem of testing hypothesis

$$H_0 : f(x) = 1, x \in (0,1)$$

versus nonparametric alternatives

$$H_n : f \in V_n = V_n(P_0) = \{f : ||f - 1|| \geq cn^{-r}, f \in U(P_0)\}$$

where $U(P_0)$ is a ball in some functional space $L \subset L_2(0,1)$. Here $r, c, c > 0, 0 < r < 1/2$, are constants and $P_0^{1/2}$ is the radius of a ball $U(P_0)$.

For this setup the same definition of maxiset and maxispace can be implemented.

Definition of χ^2-tests

Let $\hat{F}(x)$ be empirical c.d.f. of $X_1, \ldots, X_n$.
Denote $\hat{p}_{in} = \hat{F}((i+1)/k_n) - \hat{F}(i/k_n), 1 \le i \le k_n$.
Test statistics of χ^2-tests equal

$$T_n(\hat{F}_n) = k_n n \sum_{i=1}^{k_n} (\hat{p}_{in} - 1/k_n)^2$$

Theorem

For the χ^2-tests with $k_n \asymp n^{2-4r} = n^{\frac{2}{1+4s}}$ Besov spaces $B^s_{2\infty}$ with $s = \frac{r}{2-4r}$ are n^{-r}-maxispaces. Here $r = \frac{2s}{4s+1}$

Discussion

Besov spaces $B^s_{2\infty}$ does not contain stepwise functions. It seems strange. The definition of χ^2 - tests is based on indicator functions. Thus χ^2 - tests should detect well distribution functions with stepwise densities.
Let us consider χ^2 - test with $k_n = 2^{l_n}$. Then χ^2 - test statistics admit representation

$$T_n(\hat{F}_n) = k_n n \sum_{i=1}^{l_n} \sum_{j=1}^{2^i} \hat{\beta}^2_{ij}$$

with

$$\hat{\beta}_{ij} = \frac{1}{n} \sum_{m=1}^{n} \phi_{ij}(X_m)$$

where ϕ_{ij} are functions of Haar orthogonal system.

Implementing the same reasoning as in the case quadratic test statistics we get that χ^2 - test statistics have maxisets

$$\bar{B}^s_{2\infty}(P_0) = \left\{ f : f = 1 + \sum_{k=1}^{\infty} \sum_{j=1}^{2^k} \beta_{kj}\phi_{kj}, \ \sup_{\lambda>0} 2^{2\lambda s} \sum_{k>\lambda}^{\infty} \sum_{j=1}^{2^k} \beta^2_{kj} \le P_0 \right\}.$$

This statement is true as well.
Suppose function f is sufficiently smooth and β_{kj} are Fourier coefficients of f for Haar orthogonal system. Since $\beta_{kj} = 2^{-k/2} \frac{df}{dx}(j2^{-k})(1 + o(1))$ then

$$\sum_{j=1}^{2^k} \beta^2_{kj} = C2^{-k/2} \int \left(\frac{df}{dx}\right)^2 dx(1 + o(1))$$

Thus we saw that f does not belong to $B^s_{2\infty}$ for such a setup. Kernel-based tests also detect stepwise densities well. However these densities also does not belong corresponding maxispace.

Maxispaces for Cramer-von Mises tests

We shall consider Cramer- von Mises test statistics as functionals $T(\hat{F}_n - F_0)$ depending on empirical distribution function $\hat{F}_n$

$$T^2_\omega(\hat{F}_n - F_0) = \int_0^1 (\hat{F}_n(x) - F_0(x))^2 dF_0(x).$$

Theorem

The space $B^s_{2\infty}$ with $s = \frac{2r}{1-2r}$ is n^{-r}-maxispace for Kramer-von Mises test statistics. Here $r = \frac{s}{2+2s}$.

Asymptotically minimax estimators on maxisets

. For wavelet setup asymptotically minimax estimators for are wellknown $B^s_{2\infty}$ (see I.Johnstone. Gaussian Estimation: Sequence and Wavelet Models Ch 14 to be published)

$$B^s_{2\infty}(P_0) == \left\{ f : f = \sum_{k=1}^{\infty} \sum_{j=1}^{2^k} \theta_{kj}\phi_{kj}, \;\; \sup_k 2^{2ks} \sum_{j=1}^{2^k} \theta_{kj}^2 \le P_0 \right\}.$$

Asymptotically minimax tests one can find in Ingster and Suslina (Problems of Information Transmition (1998) v.34).

Minimax estimators on maxisets. Trigonometric system of functions

$$H(s, P_0) = B^s_{2\infty}(P_0) = \left\{ f : f = \sum_{j=1}^{\infty} \theta_j \phi_j,\ \ \sup_{\lambda>0} \lambda^{2s} \sum_{j>\lambda} \theta_j^2 < P_0 \right\}.$$

. The results will be provided in terms of sequence model. Let we observe a random sequence $y = \{y_j\}_{j=1}^{\infty}$ of observations

$$y_j = x_j + \epsilon\sigma_j\xi_j, \quad \epsilon > 0, \quad 1 \le j < \infty$$

where $\sigma_j > 0$ are known constants and $\xi_j, 1 \le j < \infty$, are independent Gaussian random variables $E\xi_j = 0$ and $E\xi_j^2 = 1$.
The problem is to estimate the parameter $x = \{x_j\}_{j=1}^{\infty}$.
Denote $\sigma = \{\sigma_j\}_{j=1}^{\infty}$ and $\xi = \{\xi_j\}_{j=1}^{\infty}$.
For the estimation with fixed $\epsilon > 0$ we suppose a priori information is provided in the following form

$$x \in B = B(a, P_0) = \left\{ x = \{x_i\}_{i=1}^{\infty} : \sup_k a_k^{-1} \sum_{j=k}^{\infty} x_j^2 \le P_0 \right\} \quad (1)$$

where $a = \{a_k\}_{k=1}^{\infty}$ and $a_k > 0$ is decreasing sequence.

We say that linear estimator $\hat{x}_\epsilon = \{\hat{x}_{\epsilon j}\}_{j=1}^{\infty}$ is minimax in the class of linear estimators
$\hat{x}_{\epsilon\lambda} = \{\hat{x}_{\epsilon j \lambda_j}\}_{j=1}^{\infty}, \hat{x}_{\epsilon j \lambda_j} = \lambda_j y_j, \lambda_j \in R^1, 1 \le j < \infty, \lambda = \{\lambda_j\}_{j=1}^{\infty}$, if

$$\sup_{x \in B} E_x ||\hat{x}_\epsilon - x||^2 = \inf_\lambda \sup_{x \in B} E_x ||\hat{x}_{\epsilon\lambda} - x||^2. \qquad (2)$$

The minimax estimator in the class of linear estimators will be established if the following assumptions hold.
A1 There is $c > 0$ such that $c < \sigma_j^2 < \infty$ for all j.
A2. For all $j > 1$

$$\frac{\sigma_j^2 (a_{j-1} - a_j)}{\sigma_{j-1}^2 (a_j - a_{j+1})} > 1. \qquad (3)$$

Theorem

Assume A1,A2. Then the linear estimator $\hat{\theta}_\lambda$ with

$$\lambda_j = \frac{P_0(a_j - a_{j+1})}{P_0(a_j - a_{j+1}) + \epsilon^2\sigma_j^2}. \tag{4}$$

is minimax on the set of all linear estimators.
The minimax risk equals

$$R_{l\epsilon} = \epsilon^2 \sum_{j=1}^{\infty} \frac{P_0\sigma_j^2(a_j - a_{j-1})}{P_0(a_j - a_{j-1}) + \epsilon^2\sigma_j^2}. \tag{5}$$

Asymptotically minimax estimators on maxisets.

We say that the estimator $\hat{x}_\epsilon$ is asymptotically minimax if

$$\sup_{x \in B^r_{2\infty}(P_0)} E_x||\hat{x}_\epsilon - x||^2 = \inf_{\tilde{x}_\epsilon \in \Psi} \sup_{x \in B^r_{2\infty}(P_0)} E_x||\tilde{x}_\epsilon - x||^2 (1 + o(1)) \quad (6)$$

as $\epsilon \to 0$. Here Ψ is the set of all estimators.

we replace A2 more simple assumption.
B1. For all $j > j_0$

$$\frac{\sigma_j^2 j^{2s+1}}{\sigma_{j-1}^2 (j-1)^{2s+1}} > 1. \quad (7)$$

Theorem

Assume A1,B1. Then the linear estimator $\hat{\theta}_\lambda$ with

$$\hat{\theta}_{\lambda,j} = \frac{2rP_0 j^{-2s-1}}{2sP_0 j^{-2s-1} + \epsilon^2\sigma_j^2} y_j. \tag{8}$$

is asymptotically minimax on the set of all estimators.
The asymptotically minimax risk equals

$$R_\epsilon = \epsilon^2 \sum_{j=1}^{\infty} \frac{2rP_0 j^{-2s-1}\sigma_j^2}{2sP_0 j^{-2s-1} + \epsilon^2\sigma_j^2}(1 + o(1)). \tag{9}$$

Remark

The estimator $\hat{\theta}_\lambda$ is maximum penalized likelihood estimator with quadratic penalty function

$$1/2P_0^{-1}\sum_{j=1}^{\infty} j^{2s+1}x_j^2$$

This is the standard penalty function for spline estimators. Thus spline estimators are asymptotically minimax on Besov balls $B^r_{2\infty}(P_0)$.

This estimator can be also considered as the estimator of Tikhonov regularization algorithm with corresponding regularization addendum.

Bayes approach

The asymptotically minimax estimator and Bayes estimator with a priory Gaussian measure $\theta_j = N(0, 2rP_0 j^{-2r-1})$ coincides. Here $\theta_j, 1 \le j < \infty$ are i.r.v.'s. The risks coincides as well.

If we consider asymptotically minimax estimation on ellipsoid

$$\theta \in \left\{ \theta : \sum_{j=1}^{\infty} j^{2\beta} \theta_j^2 \le P_0 \right\}, \quad b_j \to \infty$$

then asymptotically minimax risk $r_{m\epsilon}$ have the same order as Bayes risk for a priori Gaussian probability measure with $\theta_j = N(0, j^{-2\beta-1})$ (see Ermakov Inverse Problems (1990)).

For the talk setup we have the sharp equality of asymptotically minimax and Bayes risks.

Since variances σ_j^2 are not the constant the results are transfered automatically on the linear ill-posed inverse problems with random

Asymptotically minimax tests for maxisets

Our goal is to test the hypothesis

$$H_0 : \; f(t) = 0, \;\; t \in (0,1)$$

versus the alternative

$$H_\epsilon : \quad ||f||^2 > \rho_\epsilon > 0,$$

if a priori information is provided that

$$\theta \in B^r_{2\infty}(P_0) = \left\{ f : \; f(t) = \sum_{j=1}^{\infty} \theta_j \phi_j(t), \; k^{-2r} \sum_{j=k}^{\infty} \theta_j^2 \le P_0, 1 \le k < \infty \right\}$$

with $P_0 > 0$. Here $\phi_j, 1 \le j < \infty$, is orthonormal system of functions.

For wide class of orthonormal systems of functions $\phi_j, 1 \le j < \infty$ the space

$$\left\{\theta : \theta(t) = \sum_{j=1}^{\infty} \theta_j \phi_j(t),\ k^{-2r} \sum_{j=k}^{\infty} \theta_j^2 < \infty, t \in (0,1), 1 \le k < \infty\right\}$$

is Besov space $B^r_{2\infty}$

Asymptotically minimax test statistics

Define $k = k_\epsilon$ and $\kappa^2 = \kappa_\epsilon^2$ as a solution of two equations

$$2rk_\epsilon^{2r+1}\kappa_\epsilon^2 = P_0 \tag{10}$$

and

$$k_\epsilon\kappa_\epsilon^2 + k_\epsilon^{-2r}P_0 = \rho_\epsilon. \tag{11}$$

Denote $\kappa_j^2 = \kappa_\epsilon^2$, for $1 \le j \le k_\epsilon$ and $\kappa_j^2 = P_0(2r)^{-1}j^{-2r-1}$, for $j > k_\epsilon$.

Define test statistics

$$T_\epsilon^a(Y_\epsilon) = \epsilon^{-4}\sum_{j=1}^{\infty}\kappa_j^2 y_j^2.$$

$$A_\epsilon = \epsilon^{-4}\sum_{j=1}^{\infty}\kappa_j^4.$$

For type I error probabilities $\alpha, 0 < \alpha < 1$, define critical regions

$$S_\epsilon^a == \{y : (T_\epsilon^a(y) - \epsilon^{-2}\rho_\epsilon)(2A_\epsilon)^{-1/2} > x_\alpha\}$$

with x_α defined by equation

$$\alpha = 1 - \Phi(x_\alpha) = (2\pi)^{-1/2} \int_{x_\alpha}^{\infty} \exp\{-t^2/2\}\, dt.$$

Theorem

Let

$$0 < \liminf_{\epsilon \to 0} A_\epsilon \leq < \limsup_{\epsilon \to 0} A_\epsilon < \infty. \qquad (12)$$

Then the tests L_ϵ^a with critical regions S_ϵ^a are asymptotically minimax with $\alpha(L_\epsilon^a) = \alpha(1 + o(1))$ and

$$\beta_\epsilon(L_\epsilon^a) = \Phi(x_\alpha - (A_\epsilon/2)^{1/2})(1 + o(1)) \qquad (13)$$

as $\epsilon \to 0$.

Example. Let $\rho_\epsilon = R\epsilon^{\frac{8\beta}{4\beta+1}}$. Then

$$A_\epsilon = \left(\frac{P_0}{2r}\right)^{1/2r} \frac{4r+2}{4r+1} \left(\frac{R}{2r+1}\right)^{\frac{4r-1}{2r}}.$$

THANK YOU FOR ATTENTION

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