On Solving Optimal Stopping Problems

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Suppose $X = (X_s^1, X_s^2, ..., X_s^n)_{s\geq 0}$ is a strong Markov process.

We aim to find the "value" function $V^* = V^*(\cdot)$ and the optimal stopping time τ^*, such that

$$\begin{aligned} V^*(x_1, x_2, ...x_n) &= \sup_{\tau\in\mathcal{M}} \mathbf{E}_{x_1,x_2,...,x_n}\left(e^{-q\tau} g(X_\tau^1, X_\tau^2, ..., X_\tau^n)\right) \\ &= \mathbf{E}_{x_1,x_2,...,x_n}\left(e^{-q\tau^*} g(X_{\tau^*}^1, X_{\tau^*}^2, ..., X_{\tau^*}^n)\right), \end{aligned}$$

Theorem

Assume that there exists an upper semicontinuous function f, $f:\mathbb{R}^n \to R$ and $S \subset \mathbb{R}^n$ such that

(a) $f(x) \geq 0$ *for* $x \in S$, *and* $f(x) < 0$ *for* $x \notin S$.

(b)
- $\mathbf{E}_x \left(\sup_{0 \leq t \leq e_q} f(X_t) \right) = g(x)$ *for* $x \in S$,
- $\mathbf{E}_x \left(\sup_{0 \leq t \leq e_q} f(X_t) \right) \geq g(x)$ *for* $x \notin S$,

(e_q is and exponentially distributed random variable.)
Then the value function V^* *for our optimal stopping problem is*

$$V^*(x) = \mathbf{E}_x \left(\sup_{0 \leq t \leq e_q} \left(f(X_t) \; \mathbb{1}_{\{X_t \in S\}} \right) \right)$$

and the optimal stopping time $\tau^* = \inf \{ t \geq 0 : X_t \in S \}$.

Ok, we should find f to serve as an indicator where to stop.

$$S = \{x : f(x) \geq 0\}.$$

The value function V^* *is*

$$V^*(x) = \mathbf{E}_x \left(\sup_{0 \leq t \leq e_q} \left(f(X_t) \; \mathbb{1}_{\{X_t \in S\}} \right) \right)$$

the optimal stopping time $\tau^* = \inf \{t \geq 0 : X_t \in S\}$.

But how do we find f?

Consider $X = (X_s)_{s \geq 0} = (X_s^1, X_s^2, ..., X_s^n)_{s \geq 0}$, and $f : \mathbb{R}^n \to \mathbb{R}$.

Define $D_t = \{y : y = X_s, t \leq s \leq e_q\}$.

Why do we need it?

Now we can write

$$\sup_{t \leq s \leq e_q} f(X_s) = f\left(\operatorname*{argmax}_{y \in D_t} f(y)\right)$$

Definition

The Appell integral transform with respect to random variable $\nu = (\nu_1, ..., \nu_n)$ *of function* $g = g(\cdot)$ *is defined by*

$$\mathcal{A}^{\nu}\{g\}(y_1, ..., y_n) = \int_{\mathbb{R}^n} \mathcal{F}^{-1}\{g\}(u_1, ..., u_n) \frac{e^{iu_1 y_1 + ... iu_n y_n}}{\mathbf{E} e^{iu_1 \nu_1 + ... + iu_n \nu_n}} du_1 ... du_n$$

$$= \int_{\mathbb{R}^n} \mathcal{F}^{-1}\{g\}(u) \frac{e^{iuy}}{\mathbf{E} e^{iu\nu}} du,$$

where $\mathcal{F}^{-1}\{g\}$ is an inverse Fourier transform.

Averaging property of Appell Integral Transform

Lemma

$\mathbf{E}\mathcal{A}^{\nu}\{g\}(x+\nu) = g(x).$

Indeed,

$$
\begin{aligned}
\mathbf{E}\mathcal{A}^{\nu}\{g\}(x+\nu) &= \mathbf{E}\int_{\mathbb{R}^n} \mathcal{F}^{-1}\{g\}(u)\frac{e^{u(x+\nu)}}{\mathbf{E}e^{u\nu}}du \\
&= \int_{\mathbb{R}^n} \mathcal{F}^{-1}\{g\}(u)\mathbf{E}\frac{e^{u(x+\nu)}}{\mathbf{E}e^{u\nu}}du \\
&= \int_{\mathbb{R}^n} \mathcal{F}^{-1}\{g\}(u)\frac{e^{ux}\mathbf{E}e^{u\nu}}{\mathbf{E}e^{u\nu}}du \\
&= g(x)
\end{aligned}
$$

The averaging property of Appell Integral transform means we can find f in the case of Lévy processes.

Let

$$\eta = \operatorname*{argmax}_{y \in D_{\tau^+}} f(y) - x,$$

$$f(y) = \mathcal{A}^{\eta}\{g\}(y).$$

Here $\tau^+ = \inf\{t \geq 0 : x + X_t \in S\}$, $S = \{y : f(y) \geq 0\}$, $D_{\tau^+} = \{y : y = x + X_s, \tau^+ \leq s \leq e_q\}$.

- $\mathbf{E}_x\left(\sup_{0\le t\le e_q} f(X_t)\right) = g(x)$ for $x \in S$,

$$\begin{aligned}
\mathbf{E}\left(\sup_{0\le t\le e_q} f(x+X_t)\right) &= \mathbf{E}\left(\sup_{\tau^+\le t\le e_q} f(x+X_t)\right)\\
&= \mathbf{E}\left(f(x+\eta)\right)\\
&= \mathbf{E}\mathcal{A}^\eta\{g\}(x+\eta) = g(x).
\end{aligned}$$

- $\mathbf{E}_x\left(\sup_{0\le t\le e_q} f(X_t)\right) \ge g(x)$ for $x \notin S$,

$$\begin{aligned}
\mathbf{E}_x\left(\sup_{0\le t\le e_q} f(X_t)\right) &\ge \mathbf{E}_x\left(\sup_{\tau^+\le t\le e_q} f(X_t)\right)\\
&= \mathbf{E}\mathcal{A}^\eta\{g\}(x+\eta) = g(x).
\end{aligned}$$

Theorem

When X is a Levy process, the value function is given by

$$V^*(x) = \mathbf{E}_x \left(\sup_{0 \le t \le e_q} \mathcal{A}^\eta\{g\}(x + X_t) \, \mathbb{1}_{\{x + X_t \in S\}} \right)$$

and the optimal stopping time $\tau^* = \inf\{t \ge 0 : x + X_t \in S\}$, *where* $S = \{y : \mathcal{A}^\eta\{g\}(y) > 0\}$.

But we still have a problem. To construct f we need to know η, and to construct η we need to know f.

But still we have a problem. To consruct f we need to know η, and to contruct η we need to know f.

There is a special case in dimension 1, when g is a monotone function.

Appell polynomials $Q_n^\nu(y)$ are traditionally defined for one variable as

$$Q_n^\nu(y) = \frac{d^n}{du^n}\left(\frac{e^{uy}}{\mathbf{E}\left(e^{u\nu}\right)}\right)\Bigg|_{u=0} \tag{5.1}$$

in other words, $\frac{e^{uy}}{\mathbf{E}(e^{u\nu})}$ is the generating function for Appell polynomials

$$\frac{e^{uy}}{\mathbf{E}(e^{u\nu})} = \sum_{n=0}^{\infty} \frac{u^n}{n!} Q_n^\nu(y). \tag{5.2}$$

- $Q_n^{X_t}(X_t)$ are martingales, if X_t is a Lévy process,
- $\frac{d}{dy} Q_n^\nu(y) = n Q_{n-1}^\nu(y)$,
- Appell polynomials are Bell polynomials in cumulants.

$$\frac{e^{iuy}}{\mathbf{E}(e^{iu\nu})} = \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} Q_n^{\nu}(y). \tag{5.3}$$

We can generalize the construction into multiple dimensions, by taking partial derivatives of

$$\frac{e^{iu_1 y_1 + \ldots + iu_n y_n}}{\mathbf{E}\left(e^{iu_1 \nu_1 + \ldots + iu_1 \nu_1}\right)}.$$

Proposition

The Appell integral transform of the monomial y^n is the corresponding Appell polynomial $Q_n^\nu(y)$.

PROOF. By $\mathcal{F}^{-1}\{g\}$ we denote the inverse Fourier transform for some function $g = g(\cdot)$. By $\delta^{(n)}(u)$ we denote the n'th derivative of the delta function. More precisely,

$$\int_{-\infty}^{\infty} \delta^{(n)}(u)\phi(u)du = (-1)^n \phi^{(n)}(0) \tag{5.4}$$

Note, that the inverse Fourier transform of y^n is the n'th derivative of the delta function,

$$\mathcal{F}^{-1}\{y^n\}(u) = (i)^n \delta^{(n)}(u).$$

Indeed,

$$\begin{aligned}\int_{-\infty}^{\infty} \mathcal{F}^{-1}\{y^n\}(u) e^{iuy} du &= \int_{-\infty}^{\infty} i^n \delta^{(n)}(u) e^{iuy} du \\ &= (-i)^n \frac{d^n}{du^n} \left(e^{iuy}\right)\bigg|_{u=0} = y^n.\end{aligned}$$

Therefore,

$$\begin{aligned} Q^{\nu}_{y^n}(y) &= \int_{-\infty}^{\infty} \mathcal{F}^{-1}\{y^n\}(u)\frac{e^{iuy}}{\mathbf{E}e^{iu\nu}}du \\ &= \int_{-\infty}^{\infty} i^n\delta^{(n)}(u)\frac{e^{iuy}}{\mathbf{E}e^{iu\nu}}du = \\ &= (-i)^n\frac{d^n}{du^n}\left(\frac{e^{iuy}}{\mathbf{E}\left(e^{iu\nu}\right)}\right)\Bigg|_{u=0} \\ &= Q^{\nu}_n(y) \end{aligned}$$

$\triangle$

Thus with a slight abuse of notation we write for simplicity $Q^{\nu}_n(y)$ instead of $Q^{\nu}_{y^n}(y)$.

Note, that the inverse Fourier transform of $y_1^{n_1}...y_k^{n_k}$ is the corresponding derivative of the delta function,

$$\mathcal{F}^{-1}\{y^n\}(u) = i^{n_1+\ldots+n_k}\delta^{(n_1,\ldots,n_k)}(u).$$

Indeed,

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{F}^{-1}\{y_1^{n_1}...y_k^{n_k}\}(u)e^{iuy}du &= \int_{\mathbb{R}^n} i^{n_1+\ldots+n_k}\delta^{(n_1,\ldots,n_k)}(u)e^{iuy}du \\ &= (-i)^{n_1+\ldots+n_k}\,\frac{d_1^n}{du_1}\cdots\frac{d_k^n}{du_k}\left(e^{iuy}\right)\Big|_{u=0} \\ &= y_1^{n_1}...y_k^{n_k}.\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathcal{A}^{\nu}\{y_1^{n_1}\dots y_k^{n_k}\}(y) &= \int_{\mathbb{R}^n} \mathcal{F}^{-1}\{y_1^{n_1}\dots y_k^{n_k}\}(u)\frac{e^{iuy}}{\mathbf{E}e^{iu\nu}}du \\
&= \int_{\mathbb{R}^n} i^{n_1+\dots n_k}\delta^{(n_1,\dots,n_k)}(u)\frac{e^{iuy}}{\mathbf{E}e^{iu\nu}}du = \\
&= \frac{d^n}{du^n}\left(\frac{e^{iuy}}{\mathbf{E}\left(e^{iu\nu}\right)}\right)\Bigg|_{u=0}
\end{aligned}
$$

$\triangle$

Let the reward function g be given by linear combination of exponentials

$$g(y) = \sum_{k=0}^{n} c_k e^{r_k y}.$$

One can notice that the inverse Fourier transform of $e^{r_k y}$ is the delta function at $-ir_k$,

$$\mathcal{F}^{-1}\{e^{r_k y}\}(u) = \delta(u + ir_k).$$

Indeed,

$$\int_{-\infty}^{\infty} \mathcal{F}^{-1}\{e^{r_k y}\}(u) e^{iuy}\, du = \int_{-\infty}^{\infty} \delta(u+ir_k) e^{iuy}\, du = \left. e^{iuy}\right|_{u=-ir_k} = e^{r_k y}.$$

Proposition

Let $g(y) = \sum_{k=0}^{n} c_k e^{r_k y}$. Then the Appell integral transform of g is a sum of the corresponding Esscher transforms, i.e.

$$\mathcal{A}^{\nu}\{g\}(y) = Q_g^{\nu}(y) = \sum_{k=0}^{n} c_k \frac{e^{r_k y}}{\mathbf{E} e^{r_k \nu}}.$$

By analogy we extend it to the multidimensional case.

Proposition

Let $g(y) = e^{r_1 y_1 + \ldots r_k y_k}$. Then the Appell integral transform of g is

$$\mathcal{A}^{\nu}\{g\}(y) = \frac{e^{r_1 y_1 + \ldots r_k y_k}}{\mathbf{E} e^{r_1 \nu_1 + \ldots r_k \nu_k}}.$$

Let the reward function g be given by an exponential polynomial

$$g(y) = \sum_{k=0}^{n} c_k y^k e^{r_k y}.$$

Note that the inverse Fourier transform of $y^k e^{r_k y}$ is the k'th derivative of the delta function at $-ir_k$,

$$\mathcal{F}^{-1}\{y^k e^{r_k y}\}(u) = (-1)^k \delta^{(k)}(u + ir_k).$$

Indeed,

$$\begin{aligned}\int_{-\infty}^{\infty} \mathcal{F}^{-1}\{y^k e^{r_k y}\}(u) e^{iuy} du &= \int_{-\infty}^{\infty} (-1)^k \delta^{(k)}(u + ir_k) e^{iuy} du \\ &= \frac{d^k}{du^k} \left(e^{iuy}\right)\Big|_{u=-ir_k} = y^k e^{r_k y}.\end{aligned}$$

Denote the k-th derivative in u of $\frac{e^{uy}}{\mathbf{E}e^{u\nu}}$ at $u = a$ by $Q_k^\nu(y; a)$:

$$Q_k^\nu(y; a) := \frac{d^k}{du^k}\left(\frac{e^{uy}}{\mathbf{E}e^{u\nu}}\right)\Bigg|_{u=a} \tag{5.5}$$

Proposition

Let $g(y) = \sum_{k=0}^n c_k y^k e^{r_k y}$. Then the Appell integral transform of g is a

$$\mathcal{A}^\nu\{g\}(y) = Q_g^\nu(y) = \sum_{k=0}^n c_k Q_k^\nu(y; r_k).$$

%subsubsectionThe martingale property of $\mathcal{A}^{X_t}\{g\}(X_t)$ for a Lévy process X.

The linearity of $\mathcal{A}^\nu$-transform follows from linearity of the inverse Fourier transform.

Lemma

Let Appel integral transforms exist for the real functions f and g. Then

$$\mathcal{A}^\nu\{c_1 f + c_2 g\}(y) = c_1\mathcal{A}^\nu\{f\}(y) + c_2\mathcal{A}^\nu\{g\}(y), \tag{5.6}$$

where c_1 and c_2 are some constants.