Risk indicators and convergence rates for semi-Markov models

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Asymptotic Statistics of Stochastic Processes and Applications

Outline

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Motivation

Context

- How could we estimate, prevent and manage the risk of failures for repairable systems ?
- Which are the stochastic models to describe such systems ?

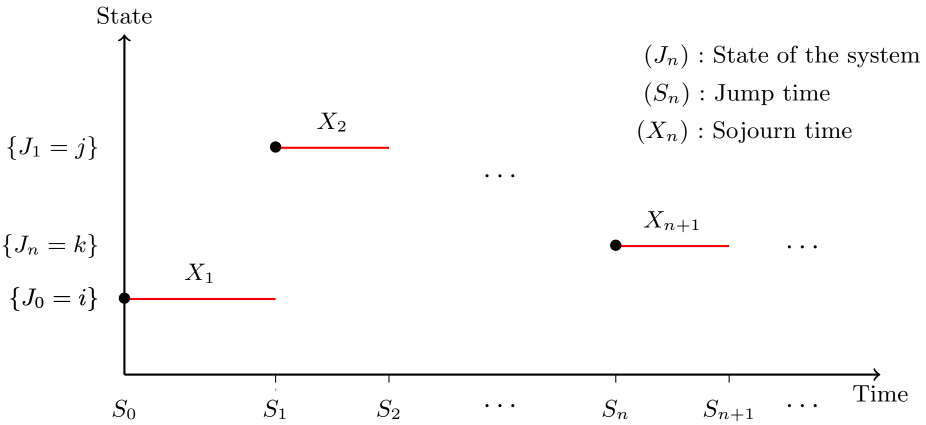
  $\hookrightarrow$ Poisson, Markov, Cox, semi-Markov, etc.

Objectives

- Describe repairable systems by semi-Markov models.
- Estimate non-parametrically risk indicators

  $\hookrightarrow$ ROCOF, reliability, availability, MTTF, MTBF, etc.

Semi-Markov models



Definition

The chain $(J,S) = (J_n, S_n)_{n\in\mathbb{N}}$ *is a Markov renewal chain (MRC) and satisfies almost surely (a.s.)* $\forall k, n \in \mathbb{N}$, $\forall i, j \in E$

$$P(J_{n+1} = j, X_{n+1} = k | J_0, ..., J_n = i; S_0, ..., S_n) = P(J_{n+1} = j, X_{n+1} = k | J_n = i).$$

Definition

We define the semi-Markov kernel (SMK) (discrete time) :

$$q_{ij}(k) \quad = \quad P(J_{n+1} = j, X_{n+1} = k | J_n = i) \quad i, j \in E, \quad k, n \in \mathbb{N}.$$

Embedded Markov chain

The chain $J = (J_n)_{n\in\mathbb{N}}$ (*embedded Markov chain*) takes its values in E and describes the state of the system in the $n-$th jump and is characterized by

- initial probabilities
$$\alpha_i = P(J_0 = i), \quad i \in E.$$
- transition probabilities
$$p_{ij} = P(J_{n+1} = j | J_n = i), \quad i, j \in E, \quad n \in \mathbb{N}.$$

Semi-Markov and double Markov chains

Definition

- *The semi-Markov chain (SMC)* $Z = (Z_k)_{k\in\mathbb{N}}$ *is defined by* $Z_k = J_{N(k)}$, *where* $N(k) = \max\{n \in \mathbb{N} | S_n \leq k\}$.
- *The sequence of the backward recurrence times is defined by* $U = (U_k)_{k\in\mathbb{N}}$, *where*

$$U_k = k - S_{N(k)}.$$

Theorem (Limnios and Oprişan, 2001)

The chain $(Z, U) = (Z_k, U_k)_{k\in\mathbb{N}}$ *is a double Markov chain with initial law* $\widetilde{a}$.

Transition law of (Z, U)

- Transition probabilities

$$\widetilde{P}\big((i,t_1),(j,t_2)\big) = P(Z_{k+1} = j, U_{k+1} = t_2 | Z_k = i, U_k = t_1),$$

$\forall\ (i,t_1),(j,t_2) \in E \times \mathbb{N},\ \forall\ k \in \mathbb{N}.$

- Survival function of sojourn times

$$\overline{H}_i(k) := P(X_{l+1} > k | J_l = i) = 1 - \sum_{j \in E} \sum_{n=0}^{k} q_{ij}(n),$$

$\forall i \in E,\ k \in \mathbb{N},\ l \in \mathbb{N}^*.$

Theorem (Chryssaphinou et al., 2008)

$$\widetilde{P}\big((i,t_1),(j,t_2)\big) = \begin{cases} q_{ij}(t_1+1)/\overline{H}_i(t_1), & \textit{if} \quad i \neq j,\ t_2 = 0, \\ \overline{H}_i(t_1+1)/\overline{H}_i(t_1), & \textit{if} \quad i = j,\ t_2 - t_1 = 1, \\ 0, & \textit{elsewhere}, \end{cases}$$

$\forall\ (i,t_1),(j,t_2) \in E \times \mathbb{N},\ \forall\ k \in \mathbb{N}.$

Risk indicators for semi-Markov models

Hitting time intensity

joint work with N.Limnios, LMAC, University of Technology of Compiègne

Hitting time intensity

Z takes its values in $E = \{1, 2, ..., s\}$. We partition $E = U \cup D$ $(U, D \neq \emptyset)$ s.t.

- $U = \{1, 2, ..., r\}$ $\hookrightarrow$ up states
- $D = \{r + 1, ..., s\}$ $\hookrightarrow$ down states
- At time k, the number of transitions of the SMC from U to D is defined by :

$$N_U(k) = \sum_{l=1}^{k} 1_{\{Z_{l-1} \in U, Z_l \in D\}}$$

Definition

The hitting time intensity in discrete time (DTIHT) is the mean number of transitions of the SMC to D at time k :

$$\widetilde{r}_U(k) = \mathbb{E}[N_U(k) - N_U(k-1)].$$

Litterature

- The DTIHT is the discrete time analogue of the *rate of occurrence of failures (ROCOF) denoted by $ro(t)$*, $t \in \mathbb{R}^+$. It can be interpreted as follows : $ro(t)\Delta(t)+o(\Delta(t))$ $(\Delta(t) \longrightarrow 0)$ is the probability that a failure, not necessarily the first, occurs in $(t, t+\Delta(t)]$.

- Markov models (continuous time)
  - Discrete state space : Lam (1997), D'Amico (2015).

- Semi-Markov models (continuous time)
  - Discrete state space : Ouhbi and Limnios (2002).
  - Continuous state space : Limnios (2012).

- (Hidden) semi-Markov models (discrete time)
  - Discrete state space : V. et al. (2014), V. and Limnios (2015).
  - Application fields : **Seismology, Finance, Biology** etc

Evaluation

Theorem

The hitting time intensity of the SMC at time k is given by

$$\widetilde{r}_U(k) \quad = \quad \sum_{i \in U} \sum_{j \in D} \sum_{m=0}^{k-1} [(\widetilde{a}\widetilde{P}^{k-1})(i,m)]\widetilde{P}\big((i,m),(j,0)\big), \quad k \in \mathbb{N}^*.$$

Non-parametric estimation

- Trajectory of the MRC $(J_n, S_n)_{n\in\mathbb{N}}$ up to arbitrary time $M \in \mathbb{N}$:

$$H(M) = (J_0, S_1, ..., J_{N(M)-1}, S_{N(M)}, J_{N(M)}, U_M).$$

Definition

The estimator of the hitting time intensity is

$$\widehat{\widetilde{r}}_U(k, M) = \sum_{i\in U}\sum_{j\in D}\sum_{m=0}^{k-1}[(\widehat{\widetilde{a}}\widehat{\widetilde{P}}_M^{k-1})(i,m)]\widehat{\widetilde{P}}_M\big((i,m),(j,0)\big),$$

where $(\widehat{\widetilde{a}}\widehat{\widetilde{P}}_M^{k-1})(i,m)$ *is the* (i,m) *element of the vector* $\widehat{\widetilde{a}}\widehat{\widetilde{P}}_M^{k-1}$, *for every* $k \in \mathbb{N}^*$.

Following Barbu and Limnios (2008), we have :

- Empirical estimator of the SMK :

$$\widehat{q}_{ij}(k,M) = \frac{1}{N_i(M)} \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j, X_n=k\}}, \;\; k \in \mathbb{N}$$

- Empirical estimator of the survival function :

$$\widehat{\overline{H}}_i(k,M) = 1 - \sum_{j \in E} \sum_{n=0}^{k} \widehat{q}_{ij}(n,M), \;\; k \in \mathbb{N}$$

- Empirical estimator of the transition probabilities :

$$\widehat{\widetilde{P}}_M\big((i,t_1),(j,t_2)\big) = \begin{cases} \widehat{q}_{ij}(t_1+1,M)/\widehat{\overline{H}}_i(t_1,M), & \text{if } i \neq j, \;\; t_2 = 0, \\ \widehat{\overline{H}}_i(t_1+1,M)/\widehat{\overline{H}}_i(t_1,M), & \text{if } i = j, \;\; t_2 - t_1 = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Consistency

Proposition

For any fixed, arbitrary $k \in \mathbb{N}^*$, $\widehat{\widetilde{r}}_U(k, M)$ is strongly consistent in the sense that

$$\widehat{\widetilde{r}}_U(k, M) \xrightarrow[M \to \infty]{a.s.} \widetilde{r}_U(k).$$

Theorem (V. and Limnios, 2015)

In the stationary, ergodic case, the distribution of the $s^2(M+1)^2$-dimensional random vector $\xi^M = (\xi^M_{(i,t_1)(j,t_2)})_{(i,t_1),(j,t_2)\in E\times T_M}$ where

$$\xi^M_{(i,t_1)(j,t_2)} = \frac{1}{\sqrt{N_{(i,t_1)}(M)}}\Big(N_{(i,t_1)(j,t_2)}(M) - N_{(i,t_1)}(M)\Big)\widetilde{P}\Big((i,t_1),(j,t_2)\Big),$$

converges, as M tends to infinity, to the normal distribution centered at the origin with covariance matrix $\Lambda = \Big(\lambda\Big(((i,t_1),(j,t_2)),((l,t_3),(r,t_4))\Big)\Big)$, where

$$\begin{aligned}&\lambda\Big(((i,t_1),(j,t_2)),((l,t_3),(r,t_4))\Big)\\ &= \delta_{(i,t_1)(l,t_3)}\Big(\delta_{(j,t_2)(r,t_4)}\widetilde{P}\big((i,t_1),(j,t_2)\big) - \widetilde{P}\big((i,t_1),(j,t_2)\big)\widetilde{P}\big((i,t_1),(r,t_4)\big)\Big)\end{aligned}$$

$T_M = \{0,1,...,M\}$, $\quad (i,t_1),(j,t_2),(l,t_3),(r,t_4) \in E\times T_M$ *and $\delta_{\boldsymbol{ij}}$ is the symbol of Kronecker, i.e, $\delta_{\boldsymbol{ij}} = 1$ if $\boldsymbol{i=j}$ and $\delta_{\boldsymbol{ij}} = 0$ if $\boldsymbol{i} \neq \boldsymbol{j}$, for all $\boldsymbol{i,j} \in E\times T_M$.*

Theorem (V. and Limnios, 2015)

Let $\big(Z_k, U_k\big)_{k\in\mathbb{N}}$ be an homogeneous, ergodic Markov chain with stationary distribution $\widetilde{\pi}$. The random vector $F = \big(f_{(i,t_1)(j,t_2)}\big)_{(i,t_1),(j,t_2)\in E\times T_M}$, where

$$f_{(i,t_1)(j,t_2)} = \sqrt{M}\Big(\widehat{\widetilde{P}}_M\big((i,t_1),(j,t_2)\big) - \widetilde{P}\big((i,t_1),(j,t_2)\big)\Big)$$

converges, as M tends to infinity, to the normal distribution $\mathcal{N}(0,\Gamma)$, where Γ is a covariance matrix of dimension $\big(s^2(M+1)^2\big)\times\big(s^2(M+1)^2\big)$, defined by

$$\Gamma = \begin{pmatrix} \frac{1}{\widetilde{\pi}(1,0)}\Lambda(1,0) & \underline{0} & \dots & \dots & \underline{0} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \frac{1}{\widetilde{\pi}(i,t_1)}\Lambda(i,t_1) & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \dots & \dots & \dots & \frac{1}{\widetilde{\pi}(s,M)}\Lambda(s,M) \end{pmatrix}.$$

Asymptotic normality

Theorem (V. and Limnios, 2015)

Let $(Z_k, U_k)_{k\in\mathbb{N}}$ *be an homogeneous, ergodic Markov chain. For any fixed, arbitrary* $k \in \mathbb{N}^*$

$$\sqrt{M}\left(\widehat{\widetilde{r}}_U(k,M) - \widetilde{r}_U(k)\right) \xrightarrow[M\to\infty]{\mathcal{L}} \mathcal{N}(0, \Phi'\Gamma\Phi'^{\top}),$$

where $\Phi : [0,1]^d \to \mathbb{R}^+$ $(d = s^2(M+1)^2)$ *is the function*

$$\begin{aligned}
&\Phi\Big(\widetilde{P}((i',m'),(j',t'));\ (i',m'),(j',t') \in E \times T_M\Big)\\
&= \sum_{i\in U}\sum_{j\in D}\sum_{m=0}^{k-1}[(\widetilde{a}\widetilde{P}^{k-1})(i,m)]\widetilde{P}((i,m),(j,0))\\
&= \sum_{i\in U}\sum_{j\in D}\sum_{m=0}^{k-1}\Big(\sum_{s\in E}\widetilde{a}(s,0)\widetilde{P}^{k-1}((s,0),(i,m))\Big)\widetilde{P}((i,m),(j,0)).
\end{aligned}$$

Simulated data

- $H(M) = (J_0, S_1, ..., J_{N(M)-1}, S_{N(M)}, J_{N(M)}, U_M)$.
- $(J_n)_{n\in\mathbb{N}}$ takes its values in $E = \{1, 2, 3, 4\}$ ($U = \{3, 4\}$,$D = \{1, 2\}$) with transition matrix
$$P = \begin{pmatrix} 0 & 0.8 & 0.2 & 0 \\ 0.9 & 0 & 0 & 0.1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
and initial law $\alpha = (1\ 0\ 0\ 0)$. We obtain a trajectory of $(J_n)_{n\in\mathbb{N}}$ in $[0, M] \subseteq \mathbb{N}$, where $M = 1000, 1500, 2000$.
- $X_{1\to3}, X_{2\to4} \sim G(0.8)$, $X_{1\to2} \sim \mathcal{W}(0.8, 1.6)$, $X_{2\to1} \sim \mathcal{W}(0.7, 1.6)$, $X_{3\to1} \sim \mathcal{W}(0.4, 0.7)$ $X_{4\to2} \sim \mathcal{W}(0.3, 0.7)$

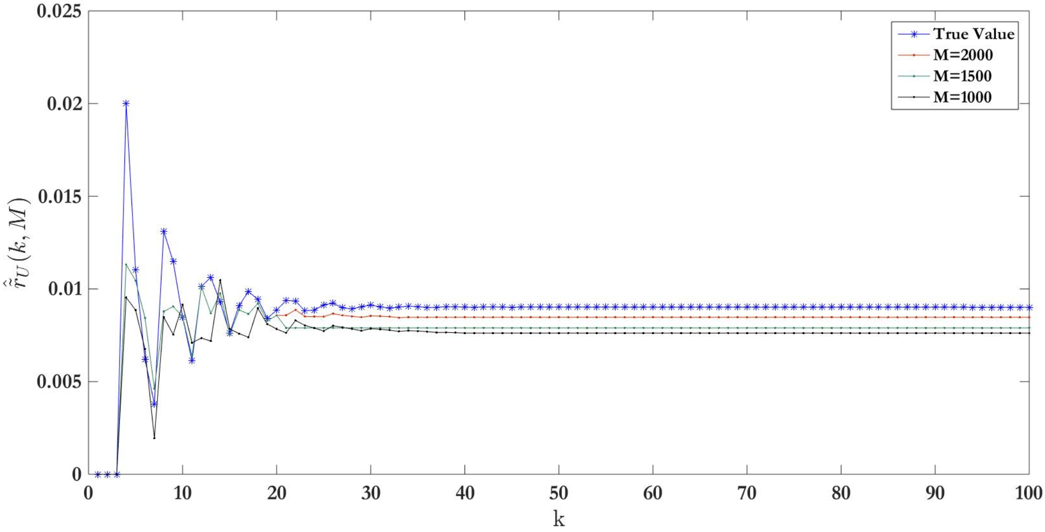


FIGURE – Consistency of the hitting time intensity estimator.

Real data

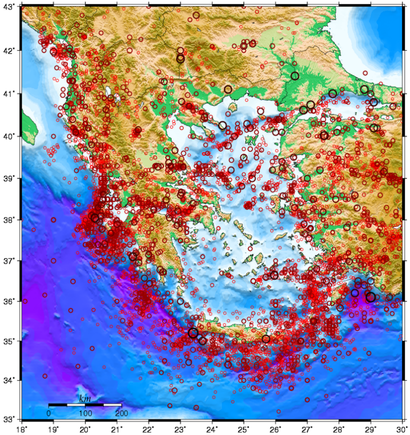


FIGURE – Epicentral distribution of earthquakes that occurred in the study area from 6^{th} century BC up to May 2011.

Data

- Study area : Greece
- Study period : $[1845, 2016]$
- Magnitudes : $M \geq 6.5$

Semi-Markov model

- $U : M \in [6.5, 7.1]$
- $D : M > 7.1$

Source

http ://geophysics.geo.auth.gr//ss

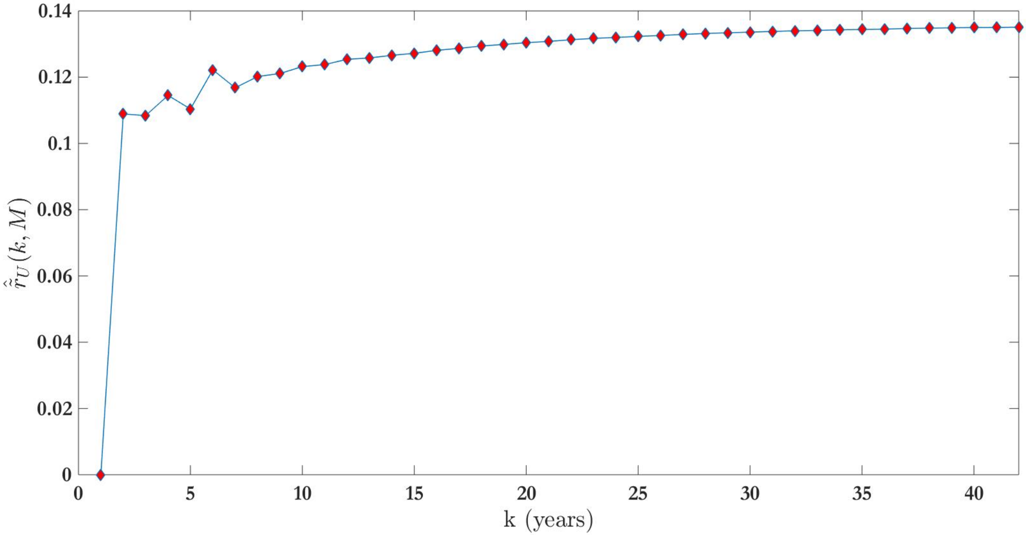


FIGURE – Occurrence rate of earthquakes with magnitudes $M > 7.1$, $\widehat{\widetilde{r}}_B(k, M)$.

Conditional hitting time intensity

joint work with M.Hamdaoui, LeM3, University of Lorraine

Conditional hitting time intensity

- At time k, the number of transitions of the SMC from the fixed state $i \in U$ to the fixed state $j \in D$ is :

$$N^{\sharp}_{ij}(k) = \sum_{l=1}^{k} \mathbf{1}_{\{Z_{l-1}=i, Z_l=j\}}.$$

Definition

The conditional hitting time intensity is the mean number of transitions of the SMC at time k given that it starts from the fixed up state i and ends to the fixed down state j :

$$r^{\sharp}_{ij}(k) = \mathbb{E}[N^{\sharp}_{ij}(k) - N^{\sharp}_{ij}(k-1)].$$

Non-parametric estimation

Definition

The estimator of the conditional hitting time intensity is defined by

$$\widehat{r_{ij}^{\sharp}}(k, M) = \sum_{m=0}^{k-1} [(\widehat{\widetilde{a}} \widehat{\widetilde{P}}_M^{k-1})(i, m)] \widehat{\widetilde{P}}_M\big((i, m), (j, 0)\big),$$

for every $k \in \mathbb{N}^*$.

Lemma

For $n \geq 2$, the random vector $F^n = \left(f^n_{(i',m'),(j',t')}\right)_{(i',m'),(j',t') \in E \times T_M}$ where

$$f^n_{(i',m'),(j',t')} = \sqrt{M}\left(\widehat{\widetilde{P}}^n\left((i',m'),(j',t')\right) - \widetilde{P}^n\left((i',m'),(j',t')\right)\right)$$

has the same limit in distribution as the random vector

$$G^n = \left(g^n_{(i',m'),(j',t')}\right)_{(i',m'),(j',t') \in E \times T_M},$$

where

$$g^n_{(i',m'),(j',t')} = \sum_{(j,t) \in E \times T_M} \left(\widetilde{P}^{n-1}\left((i',m'),(j,t)\right) f_{(j,t),(j',t')} + \widetilde{P}^{n-1}\left((j,t),(j',t')\right) f_{(i',m'),(j,t)}\right)$$
$$+ \mathbf{1}_{\{n \geq 3\}} \sum_{k=2}^{n-1} \sum_{(i_1,t_1),(i_2,t_2) \in E \times T_M} \widetilde{P}^{n-k}\left((i',m'),(i_1,t_1)\right) \widetilde{P}^{k-1}\left((i_2,t_2),(j',t')\right) f_{(i_1,t_1),(i_2,t_2)}.$$

Lemma

The random vector F^n converges, as M tends to infinity, to a centered normal random vector with covariance matrix $\Sigma = \Sigma_f \, \Gamma \, \Sigma_f^T$. The matrix

$$\Sigma_f = \Sigma_f((i',m'),(j',t'),(u',v'),(s',t'))_{(i',m'),(j',t'),(u',v'),(s',w') \in E \times T_M}$$

of dimension $d \times d$ ($d = s^2(M+1)^2$) is given by

$$\begin{aligned}\Sigma_f((i',m'),(j',t'),(u',v'),(s',w')) = {} & \delta_{(j',t'),(s',w')} \, \widetilde{P}^{n-1}\big((i',m'),(u',v')\big) \\ & + \delta_{(i',m'),(u',v')} \, \widetilde{P}^{n-1}\big((s',w'),(j',t')\big) \\ & + \sum_{k=2}^{n-1} \widetilde{P}^{n-k}\big((i',m'),(u',v')\big) \, \widetilde{P}^{k-1}\big((s',w'),(j',t')\big).\end{aligned}$$

Theorem

For any fixed $k \geq 3$, *the random vector* $\mathcal{R}(k) = (\mathcal{R}_{ij}(k))_{i,j \in E}$ *where*

$$\mathcal{R}_{ij}(k) = \sqrt{M}\big(\widehat{r^{\sharp}_{ij}}(k, M) - r^{\sharp}_{ij}(k)\big)$$

converges in distribution, as M *tends to infinity, to a centered normal random vector with covariance matrix* $\Sigma_r \Gamma {\Sigma_r}^{\top}$, *where*

$$\begin{aligned}
\Sigma_r((i,j),(u',v'),(s',w')) &= \delta_{(u',i)} \mathbf{1}_{(v' \leq k-1)}\, \delta_{(s',w'),(j,0)} \sum_{s \in E} \widetilde{a}(s,0) \widetilde{P}^{k-1}\big((s,0),(u',v')\big) \\
&+ \delta_{(s',i)} \mathbf{1}_{(w' \leq k-1)}\, \widetilde{P}\big((i,w'),(j,0)\big) \sum_{s \in E} \widetilde{a}(s,0) \widetilde{P}^{k-2}\big((s,0),(u',v')\big) \\
&+ \delta_{(u',v'),(s,0)}\, \widetilde{a}(u',0) \sum_{m=0}^{k-1} \widetilde{P}\big((i,m),(j,0)\big) \widetilde{P}^{k-2}\big((s',w'),(i,m)\big) \\
+ \sum_{m=0}^{k-1} \sum_{s \in E} \sum_{r=2}^{k-2} & \widetilde{a}(s,0)\, \widetilde{P}\big((i,m),(j,0)\big)\, \widetilde{P}^{k-1-r}\big((s,0),(u',v')\big)\, \widetilde{P}^{r-1}\big((s',w'),(i,m)\big).
\end{aligned}$$

Conditional mean time to failure

joint work with A.Brouste, LMM, Le Mans University

Conditional mean time to failure

Z takes its values in $E = \{1, 2, ..., s\}$. We partition $E = U \cup D$ $(U, D \neq \emptyset)$ s.t.

- $U = \{1, 2, ..., r\}$ $\hookrightarrow$ up states
- $D = \{r + 1, ..., s\}$ $\hookrightarrow$ down states
- The first passage time in D is defined by :

$$T_D = \inf\{k \in \mathbb{N} : Z_k \in D\} \quad \text{and} \quad \inf\{\varnothing\} = \infty.$$

Definition

The conditional mean time to failure is defined by :

$$CMTTF_i = \mathbb{E}(T_D | J_0 = i),$$

for any state $i \in U$.

Evaluation

Column vector of the conditional mean times to failure :

$$\begin{aligned}\mathbf{CMTTF} &= (CMTTF_1, \ldots, CMTTF_r)^\top \\ &= (\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{m_1},\end{aligned}$$

where $\mathbf{P}_{11} = (p_{ij};\ i,j \in U)$ and $\mathbf{m}_1 = (\mathbb{E}(S_1|J_0 = i);\ i \in U)$.

Non-parametric estimation

Empirical estimator of **CMTTF** :

$$\widehat{\mathbf{CMTTF}}(M) = (\mathbf{I} - \widehat{\mathbf{P}}_{11}(M))^{-1}\widehat{\mathbf{m}}_1(M),$$

where

- $\widehat{\mathbf{P}}_{11}(M) = (\widehat{p}_{ij}(M);\ i,j \in U)$ and $\widehat{p}_{ij}(M) = \frac{N_{ij}(M)}{N_i(M)}$;
- $\widehat{\mathbf{m}}_1(M) = (\widehat{m}_i(M);\ i \in U)^\top$ and $\widehat{m}_i(M) = \sum_{\ell \geq 0} \widehat{\overline{H}}_i(\ell, M)$.

Consistency

Theorem

For any state $i \in U$, $C\widehat{MTTF}_i(M)$ is strongly consistent, in the sense that

$$C\widehat{MTTF}_i(M) \xrightarrow[M\to\infty]{a.s.} CMTTF_i.$$

Asymptotic normality

Theorem

For any state $i \in U$, the random variable $\widehat{CMTTF}_i(M)$, is asymptotically normal, in the sense that

$$\sqrt{M}(\widehat{CMTTF}_i(M) - CMTTF_i) \xrightarrow[M\to\infty]{\mathcal{L}} \mathcal{N}(0, \sigma^2_{CMTTF_i}),$$

with the asymptotic variance

$$\sigma^2_{CMTTF_i} = \sum_{m\in E} a^2_{im}\mu_{mm}\Big(\sigma^2_m + \sum_{\ell\in E}(\eta_\ell - \widetilde{\eta}_m)^2 p_{m\ell} + 2\sum_{\ell\in E}\eta_\ell Q_{m\ell}\Big),$$

where $Q_{m\ell} = \sum_{u=1}^{+\infty}(u - m_m)q_{m\ell}(u)$, $a_{ij} = (\mathbf{I} - \mathbf{P}_{11})^{-1}_{ij}$, $\eta_\ell = \sum_{r\in U} m_r a_{r\ell}$, $\widetilde{\eta}_m = \sum_{j\in U} p_{mj}\eta_j$, and σ^2_m is the variance of the sojourn time in state m.

Convergence rate of the posterior distribution

joint work with G.Gayraud and N.Limnios, LMAC, UTC

Definition

- $(J_n)_{n\in\mathbb{N}}$ *is defined in a continuous space* E *with associated* σ*-algebra* $\mathcal{E}$ *;* $(S_n)_{n\in\mathbb{N}}$ *is defined in* $\mathbb{R}^+$ *with associated* σ*-algebra* $\mathcal{B}^+ = Bor(\mathbb{R}^+)$.

- *The* *Markov renewal process* $(J_n, S_n)_{n\in\mathbb{N}}$ *satisfies a.s.*

$$P(J_{n+1} \in B, S_{n+1} - S_n \le t | J_0, \ldots, J_n, S_0, \ldots, S_n) = P(J_{n+1} \in B, S_{n+1} - S_n \le t | J_n),$$

$\forall n \in \mathbb{N}$, $\forall t \in \mathbb{R}^+$ *and* $\forall B \in \mathcal{E}$.

- *The* *semi-Markov process* *is defined by* $Z_t = J_{N(t)}$, $t \in \mathbb{R}^+$, *where*

$$N(t) = \begin{cases} 0, & \text{if} \quad S_1 - S_0 > t, \\ \sup\{n \in \mathbb{N}^* : S_n \le t\}, & \text{if} \quad S_1 - S_0 < t. \end{cases}$$

Definition

Semi-Markov kernel (continuous time) :

$$Q_x(B,t) = P(J_{n+1} \in B, S_{n+1} - S_n \leq t | J_n = x),$$

$\forall B \in \mathcal{E}$, $\forall x \in E$, $t \in \mathbb{R}^+$.

Hypotheses

- (H1) There exists a measure ν in $(E \times \mathbb{R}^+, \mathcal{E} \otimes \mathcal{B}^+)$ s.t. $\forall x \in E$

$$dQ_x(y,t) = q_x(y,t) d\nu(y,t),$$

  where $q_x(\cdot,\cdot)$ is a probability density.
- (H2) $(J_n, S_n)_{n\in\mathbb{N}}$ has the stationary law $\widetilde{\rho}$.

Bayesian framework

- We assume that the probability density $q_x(\cdot,\cdot)$ is parameterized by $\vartheta \in \Theta \hookrightarrow q^{\vartheta}$.
- The parameter ϑ is random with a prior distribution π defined in $(\Theta, \mathcal{T})$.
- We consider a trajectory of the MRP

$$H_n = (J_0, S_1, ..., J_n, S_n),$$

that corresponds to ϑ, i.e. to the probability density q^{ϑ}.

Definition

The *posterior distribution* π^{H_n} *is defined by*

$$\pi^{H_n}(B) \quad = \quad \frac{\int_B \mathbb{P}_\vartheta^{(n)}(dH_n)d\pi(\vartheta)}{\int_\vartheta \mathbb{P}_\vartheta^{(n)}(dH_n)d\pi(\vartheta)},$$

$\forall B \in \mathcal{T}$, *where* $\mathbb{P}_\vartheta^{(n)}$ *is the distribution of* H_n.

Semi-Markov framework

$$\pi^{H_n}(B) = \frac{\int_B \widetilde{\rho}(J_0, S_0) \prod_{l=1}^n q_{J_{l-1}}^{\vartheta}(J_l, S_l - S_{l-1})d\pi(\vartheta)}{\int_\vartheta \widetilde{\rho}(J_0, S_0) \prod_{l=1}^n q_{J_{l-1}}^{\vartheta}(J_l, S_l - S_{l-1})d\pi(\vartheta)}$$

Convergence rate

Convergence rate

How fast the posterior distribution shrinks towards the true parameter value ϑ_0 ?

Definition

π^{H_n} *converges with a rate* ε_n*, if there exists a strictly positive sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *tending to* 0 *s.t.*

$$\pi^{H_n}\left(V^{\complement}_{\varepsilon_n}(\vartheta_0)\right) \xrightarrow[n\to\infty]{\mathbb{P}^{(n)}_{\vartheta_0}} 0,$$

where $V_{\varepsilon_n}(\vartheta_0) = \{\vartheta \,|\, d(q^{\vartheta}, q^{\vartheta_0}) \leq \varepsilon_n\}$*, and* d *a distance defined in* $Q = \{q^{\vartheta}|\vartheta \in \vartheta\}$.

Litterature

$H_n \hookrightarrow$ i.i.d. observations

Ghosal et al. (2000) ; Arbel et al. (2014) ; Castillo (2014) etc

$H_n \hookrightarrow$ non i.i.d. observations

Ghosal and van der Vaart (2007) ; Rousseau et al. (2012) ; Gassiat and Rousseau (2013) ; Knapik and Salomond (2014) etc

$H_n \hookrightarrow$ realization of a MRP

How could we obtain the convergence rate of π^{H_n} ? Under which hypotheses ?

... which distance d in Q ?

- Hellinger distance between the distributions $q_{\vartheta_1;x}$ and $q_{\vartheta_2;x}$ dominated by the measure ν and for fixed $x \in E$:

$$h_\nu^2(q_{\vartheta_1;x}, q_{\vartheta_2;x}) = \frac{1}{2}\int_{E\times\mathbb{R}^+}\left(\sqrt{q_x^{\vartheta_1}(y,t)} - \sqrt{q_x^{\vartheta_2}(y,t)}\right)^2 d\nu(y,t).$$

- Given a measure μ, we define the semi-distance d_μ between q^{ϑ_1} and q^{ϑ_2} :

$$d_\mu^2(q^{\vartheta_1}, q^{\vartheta_2}) = \int_E h_\nu^2(q_{\vartheta_1;x}, q_{\vartheta_2;x})d\mu(x).$$

For two distributions $\mathbb{P}_{\vartheta_0}^{(n)}$ and $\mathbb{P}_{\vartheta}^{(n)}$, we define :

- $\mathbb{E}_0^{(n)}$ and $\mathbb{V}_0^{(n)}$ the mean and the variance w.r.t. the distribution $\mathbb{P}_{\vartheta_0}^{(n)}$.
- Kullback-Leibler divergence :

$$K(\mathbb{P}_{\vartheta_0}^{(n)}, \mathbb{P}_{\vartheta}^{(n)}) = \mathbb{E}_0^{(n)}\Big[\log \frac{\widetilde{\rho}(J_0, S_0)}{\widetilde{\rho}(J_0, S_0)} \prod_{l=1}^{n} \frac{q_{J_{l-1}}^{\vartheta_0}(J_l, S_l - S_{l-1})}{q_{J_{l-1}}^{\vartheta}(J_l, S_l - S_{l-1})}\Big].$$

- Second central moment :

$$V(\mathbb{P}_{\vartheta_0}^{(n)}, \mathbb{P}_{\vartheta}^{(n)}) = \mathbb{V}_0^{(n)}\Big[\log \frac{\widetilde{\rho}(J_0, S_0)}{\widetilde{\rho}(J_0, S_0)} \prod_{l=1}^{n} \frac{q_{J_{l-1}}^{\vartheta_0}(J_l, S_l - S_{l-1})}{q_{J_{l-1}}^{\vartheta}(J_l, S_l - S_{l-1})}\Big].$$

Additional hypotheses

- (H3) There exist two measures ν^* and η^* in $\mathcal{E}$ and two positive integers k, l s.t. $\forall x \in E$, and all $\vartheta \in \vartheta$,

$$\nu^*(\cdot) \leq \frac{1}{k} \sum_{u=1}^{k} P_{\vartheta}^{(u)}(x, \cdot) \qquad \text{and} \qquad \eta^*(\cdot) \geq P_{\vartheta}^{(l)}(x, \cdot).$$

- (H4) $\exists\ c > 0$ e.t.

$$\pi\Big(\vartheta : K(\mathbb{P}_{\vartheta_0}^{(n)}, \mathbb{P}_{\vartheta}^{(n)}) < n\varepsilon_n^2, V(\mathbb{P}_{\vartheta_0}^{(n)}, \mathbb{P}_{\vartheta}^{(n)}) < n\varepsilon_n^2\Big) > e^{-cn\varepsilon_n^2}.$$

- (H5) For $0 < \xi < 1$, there exists $\varepsilon_n > 0$ s.t.

$$\sup_{\varepsilon > \varepsilon_n} \log N\big(\varepsilon\xi, \{\vartheta \in \vartheta : d_{\nu*}(q^{\vartheta}, q^{\vartheta_0}) \leq \varepsilon\}, d_{\eta*}\big) \leq n\varepsilon_n^2.$$

Main result

Theorem

Under the hypotheses (H1)-(H5), and for M large enough, there exists $(\varepsilon_n > 0)$ where $\varepsilon_n \longrightarrow 0$ s.t.

$$\mathbb{E}_0^{(n)}\left[\pi^{H_n}\left(\vartheta : d_{\nu^*}(q^{\vartheta}, q^{\vartheta_0}) > M\varepsilon_n\right)\right] \xrightarrow[n\to\infty]{} 0.$$

Perspectives

Risk indicators $\longrightarrow$ hitting time intensity

- SMCs in general state space
- Hidden Markov renewal chains
  - Jump times $\neq$ emission times ?

Convergence rate of the posterior distribution

- How could we weaken some of the hypotheses ?

Semi-Markov processes/chains $\longrightarrow$ parametric context

- MLEs or Bayesian estimators with appealing asymptotic properties ?

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Thank you