Overfitting of Hurst estimators for multifractional Brownian motion. A fitting test advocating simple models.

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Consequences of the Data Deluge (for series)

- Large or big datasets are described by stochastic models, more and more elaborated.
- Fractional Brownian motion are described by one parameter, the Hurst index H.
- The Hurst index H correspond to the properties :
  1. **Long memory**.
  2. statistical **self-similarity**.
  3. **Roughness** = path regularity.
  4. **Stickiness** = persistence, anti-persistence or independence of the increments.
- In behavioral finance, turbulence, biology, physics, health and medicine, the Hurst index is varying with the time.

Historical recall on some stochastic models

Stochastic model tries to better fit real datasets

- Brownian motion $(H = 1/2)$
  Einstein 1905, Bachelier 1901, Wiener 1930 ...
- Fractional Brownian motion $(0 < H < 1)$
  Kolmogorov 1940, Mandelbrot 1968.
- Multifractional Brownian motion ($H(t)$ is time-varying)
  Benassi, Jaffard, Roux 1997, Peltier, Levy-Vehel 1996, ...
- Different generalisations motivated by specific applications
  Many references since 2000.

Future ?

Stochastic model tries to better fit real datasets

- Brownian motion $(H = 1/2)$
- Fractional Brownian motion $(H \neq 1/2)$
- Multifractional Brownian motion ($H(t)$ is time-varying)
- Different generalisations motivated by specific applications

Future ?

Stochastic model tries to better fit real datasets

- Brownian motion $(H = 1/2)$
- Fractional Brownian motion $(H \neq 1/2)$
- Multifractional Brownian motion ($H(t)$ is time-varying)
- Different generalisations motivated by specific applications

What next ?

- Multifractional Brownian motion with a Hurst index $H(t,\omega)$ being itself a stochastic process ?
- A parcimonious model ?

A statistical artifact (Fractlab)

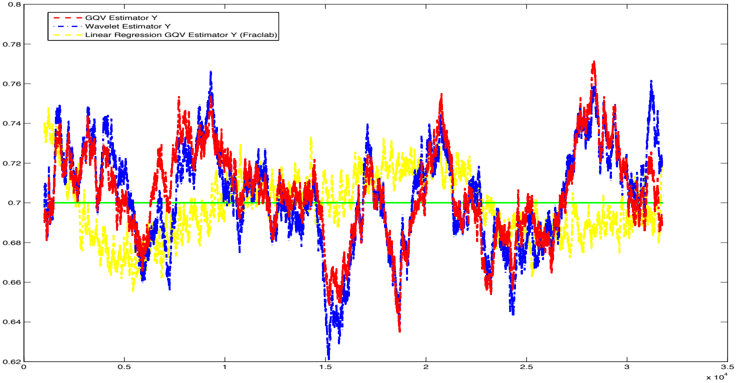

FIGURE – We have simulated a fBm with constant Hurst index $H = 0.7$ and estimated a time-varying Hurst index $\widehat{H}(t)$ with 3 different estimators (Wavelet, Quadratic Variation, Linear regression GQV)

A statistical artifact, JM Bardet (*Software for estimating the Hurst function H of a mBm*)

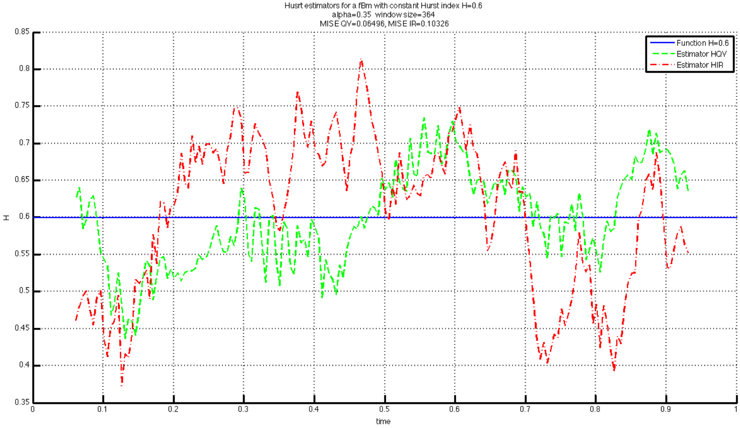

FIGURE – We have simulated a fBm with constant Hurst index $H = 0.7$ and estimated a time-varying Hurst index $\widehat{H}(t)$ with 2 different estimators (Linear regression GQV and Increment Ratio)

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Recall on fractional Brownian motion

Definition

The fractional Brownian motion (fBm), with Hurst index H and variance σ^2, is a zero mean Gaussian process with covariance

$$\begin{aligned} R_H(t_1, t_2) &= \operatorname{cov}\big(X(t_1), X(t_2)\big) \\ &= \frac{1}{2}\sigma^2 \left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}. \end{aligned}$$

- The Hurst index $H \in]0, 1[$.
- When $H = 1/2$ et $\sigma = 1$, $B_{1/2}$ is a standard Brownian motion.

Fractional Brownian motion

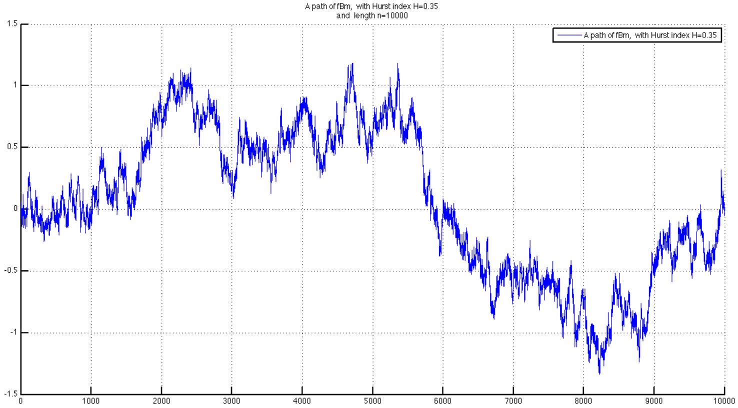


FIGURE – We have simulated a path of fBm with constant Hurst index $H = 0.35$

Fractional Brownian motion

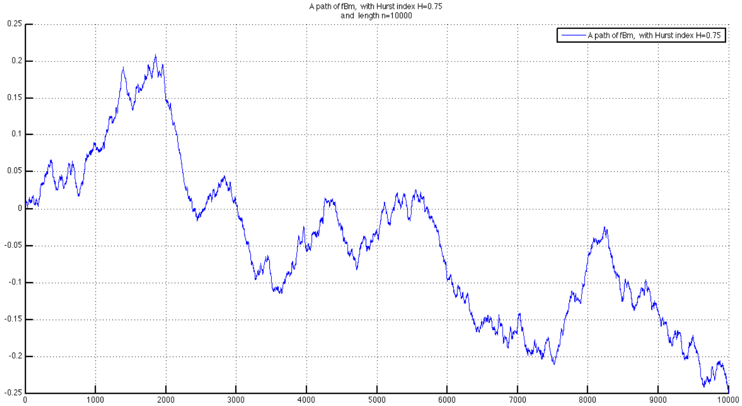


FIGURE – We have simulated a path of fBm with constant Hurst index $H = 0.75$

The Hurst index H drives 3 properties :

1. **Pathwise regularity** $\forall t,\ \alpha^*(t) = H$ *a.s.* where

$$\alpha^*(t) = \sup\left\{\alpha, limsup_{h\to 0}\frac{|X(t+h)-X(t)|}{h^\alpha} = 0\right\}$$

2. **Self-similarity :**

$$\left(B_H(\lambda t)\right)_{t\in\boldsymbol{R}} \stackrel{(d)}{=} \left(\lambda^H B_H(t)\right)_{t\in\boldsymbol{R}}.$$

3. **Correlation of the increments :**

$$r(n) = cov\left(X(n+1)-X(n), X(1)-X(0)\right).$$

If $H > 1/2$, then the paths have

- **Long memory :** $\sum_{k=-\infty}^{+\infty}|r(k)| = \infty$
- **Stickiness :** $r(1) > 0$ = the increments are positively correlated

Three representations of fBm

1. **Moving average representation** (Mandelbrot & Van Ness, 1968)

$$B_H(t) \quad = \quad C \int_{-\infty}^{+\infty} \left[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right] dW_s.$$

2. **Harmonisable representation** (Kolmogorov, 1940)

$$B_H(t) \quad = \quad \int_{\mathbb{R}} \left(e^{it\xi} - 1 \right) \times |\xi|^{-(H+1/2)} \, \widehat{W}(d\xi)$$

where $\widehat{W}(d\xi)$ is the Fourier transform of the Wiener measure $W(dx)$.

Wavelet series expansion of fBm

1. **Wavelet series expansion** (Meyer, Sellan, Taqqu, 1999)

$$B(t,H) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-jH} \varepsilon_{j,k} \left\{ \phi(2^j t - k, H) - \phi(-k, H) \right\}, \qquad (1)$$

- where $\left(\varepsilon_{j,k}\right)_{(j,k)\in\mathbf{Z}^2}$ is a family of independent Gaussian random variables $\mathcal{N}(0,1)$;
- $\{2^{j/2}\psi(2^j x - k) : (j,k) \in \mathbf{Z}^2\}$ is a Lemarié-Meyer wavelet basis ;
- and

$$\phi(x,H) = \int_{\boldsymbol{R}} e^{ix\xi} \frac{\widehat{\psi}(\xi)}{|\xi|^{H+1/2}} \, d\xi. \qquad (2)$$

The convergence of the series is uniform on every compact subset $I \times K \subset (0,1) \times \boldsymbol{R}$, almost surely (Ayache & Taqqu, 2003).

Recall on multifractional Brownian motion

Definition

- The multifractional Brownian motion (mBm) can be seen as a generalisation of the fBm
- The Hurst index $0 < H < 1$ is replaced by a time-varying function $t \mapsto H(t)$

$$X(t) = B(t, H(t))$$

where $B(t, H) := B_H(t)$ is the wavelet series expansion of fBm, or another representation.

Multifractional Brownian motion, $H(t)$ smooth

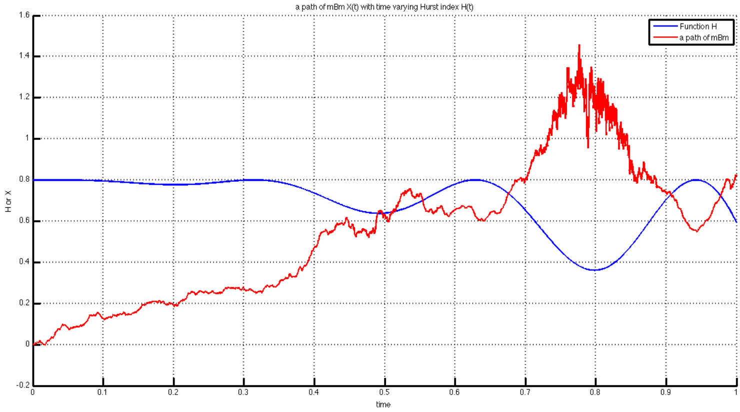


FIGURE – A path of mBm with Hurst index $H(t) = 0.1 + 0.7 \times \left[1 - t \cdot \sin^2(10t)\right]$, Software by JM Bardet

Multifractional Brownian motion, $H(t)$ rough

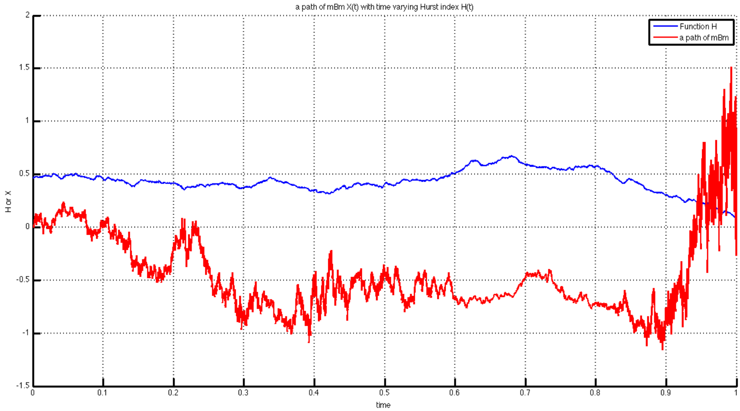


FIGURE – A path of mBm with Hurst index $H(t)$ being a fBm with $H = 0.75$, Software by JM Bardet

Applications in many fields

Models with a time-varying Hurst index can be encountered in many different fields

- In turbulence (see Papanicolaou and Solna, 2002) : the mBm with a regularly time-varying Hurst index is used for the air velocity.
- In statistical study on magnetic dynamics (see Wanliss and Dobias, 2007) : an abrupt change in Hurst index can be observed before a space storm in solar wind.

Behavioural economics...

Economic point of view is developed by Bianchi (2005) – Bianchi, Pantanella, Pianese (2015).

- Periods with Hurst index $H \neq 1/2$ can be explained by behavioural economics :
  1. $H = 1/2$ [independence of the increments] :
     = efficiency of the market.
  2. $H(t) < 1/2$ [increments negatively correlated] :
     the market is not confident in the past and it overreacts to new information.
  3. $H(t) > 1/2$ [increments positively correlated] :
     the market is too confident in the past and it underreacts to new information.
- In behavioural finance, under-reaction is due to overconfidence of investors.

. . .against mainstream Finance

- Arbitrage opportunity for fBm is possible when the Hurst index H is constant without transaction costs (Rogers 1997, Shyriaev 1998, Cheridito 2003).
- However, arbitrage with fBm does no more exist with small transaction costs (Guasoni, 2006, Guasoni, Rasonyi, Schachermayer 2010).
- Today research in finance investigates pricing for fBm, see *"Shadow prices, fractional Brownian motion, and portfolio optimisation under transaction costs"*
  Czichowsky, Peyre, Schachermayer, Yang (2016)

Estimating Hurst index for fBm

- Let X be a fBm. We observe one path of size n of the process X with mesh $h_n = \frac{1}{n}$, namely $\big(X(0), X(t_1), \ldots, X(t_n)\big)$.
- The standard method for estimating Hurst index for fBm are
  - Wavelet coefficients $\widehat{H}^{(W)}$ (Abry et al. 2003).
  - Linear Regression of Generalized Quadratic Variation $\widehat{H}^{(GQV)}$ (Benassi-Cohen-Istas, 1998, Coeurjolly, 2005)
  - Increment Ratio statistics $\widehat{H}^{(IR)}$, (Bardet-Surgailis, 2010).

Estimating Hurst index for mBm

- Let X be a fBm or a mBm. We observe one path of size n of the process X with mesh $h_n = \frac{1}{n}$, namely $\big(X(0), X(t_1), \ldots, X(t_n)\big)$.

- The standard method for estimating a time-varying Hurst index for mBm is to localise the estimation of a constant Hurst index on a small vicinity of each time t, namely on

$$\mathcal{V}(t, \varepsilon_n) = \{t_k \text{ such that } |t_k - t| \leq \varepsilon_n\},$$

where $\varepsilon_n = n^{-\alpha}$, with $0 < \alpha < 1$. Thus

$$\varepsilon_n \to 0 \qquad \textit{and} \qquad \frac{\varepsilon_n}{h_n} \to \infty \quad \textit{as} \quad n \to \infty.$$

Pointwise convergence of time varying Hurst index $H(t)$ for mBm

Localized Hurst index estimator converge for all $t \in (0,1)$ for Wavelet estimator, GQV or IR estimator
Abry, Flandrin, Taqqu, Veitch (2003),Coeurjolly (2005), Bardet-Surgailis (2013), . . .

Functional Central Limit Theorem

Coeurjolly (2005) states a functional CLT for the GQV estimator

Theorem : Coeurjolly, 2005 – 2006

If $t \longmapsto H(t)$ is regular enough, then for all time t, $H_n^{(GQV)}(t) \longrightarrow H(t)$ and

$$\sqrt{2\varepsilon_n \cdot n} \times \left(\widehat{H}_n^{(GQV)}(t) - H(t)\right) \longrightarrow_{(\mathcal{L})} \mathbb{G}'(t)$$

where $\mathbb{G}'(t)$ a zero mean Gaussian process, with covariance structure :

$$\begin{aligned} \operatorname{var}(\mathbb{G}'(t)) &= \gamma(H(t)) \quad \textit{for all } t \in (0,1), \\ \operatorname{cov}(\mathbb{G}'(t_1), \mathbb{G}'(t_2)) &= 0 \qquad \textit{with } (t_1, t_2) \in (0,1)^2 \quad \textit{for } t_1 \neq t_2, \end{aligned}$$

Complicated formulas for $\gamma^{(GQV)}(H)$

$$\gamma^{(GQV)}(H) = \left(\frac{1}{\pi_H^a(0)^2} \sum_{k \in \mathbf{Z}} \pi_H^a(k)^2 \right) \times \frac{A^t (UU^t) A}{4 \|A\|^4}$$

with

$$\begin{aligned}
\pi_H^a(k) &:= -\frac{1}{2} \sum_{q=0}^{2} \sum_{q'=0}^{2} a_q \, a_{q'} \, |q - q' + k|^{2H}, \\
a &= (1,\ -2,\ 1) \\
U &= (1, \ldots, 1)
\end{aligned}$$

and

$$A_j = \ln(j) - \frac{1}{M} \sum_{\nu=1}^{M} \ln(\nu) \text{ for all } j = 1, \ldots, M$$

To sum up, we can compute $\gamma^{(GQV)}(H)$

Functional CLT, Bardet-Surgailis (2013)

Theorem : Th.2 and Th.3 in Bardet-Surgailis, 2013

- Under some technical assumptions
- the functional CLT of Coeurjolly (2005) holds for both GQV and IR estimator, with two different limit processes $\mathbb{G}^{(GQV)}(t)$ and $\mathbb{G}^{(IR)}(t)$.
- For all family $t_1, \ldots, t_k$, $\mathbb{G}^{(IR)}(t_i)$ are independent centered Gaussian r.v.'s such as,

$$\boldsymbol{E}\left[\mathbb{G}^{(IR)}(t_i)\right]^2 = \gamma^{(IR)}(H(t)).$$

- and $\mathbb{G}^{(GQV)}(t_i)$ are independent centered Gaussian r.v.'s such as,

$$\boldsymbol{E}\left[\mathbb{G}^{(GQV)}(t_i)\right]^2 = \gamma^{(GQV)}(H(t)).$$

Recall on Increment Ratio Statistics (IRS)

Let X be observed at the discrete times $t = 1, \dots, n$.

1. We define the increments of order $L = 1$ by
$$\delta_1(t) = X_{t+1} - X_t$$
2. We define the increments of order $L = 2$ by
$$\delta_2(t) = X_{t+2} - 2X_{t+1} + X_t$$

Then, the IRS is given by

$$IRS_{L,n}(X) = \frac{1}{(n-L)} \sum_{t=1}^{n-L-1} \psi\left(\delta_L X_t, \delta_L X_{t+1}\right)$$

with

$$\psi(x,y) := \begin{cases} \dfrac{|x+y|}{|x|+|y|} & \text{if } (x,y) \in \boldsymbol{R}^2 \backslash \{(0,0)\} \\ 1 & \text{if } (x,y) = (0,0). \end{cases}$$

Central Limit Theorem (CLT) for IRS

When X is a fBm with Hurst index H, that is $X = B_H$, we have the CLT (Bardet, Surgailis, 2011) :

$$\sqrt{n}\left(IRS_{L,n}(B_H) - \lambda_0\left(\rho_L(H)\right)\right) \to \mathcal{N}(0, \sigma_L^2(H))$$

for $H \in (0,1)$ if $L = 2$ and $H \in (0, 3/4)$ for $L = 1$.
With

$$\lambda_0(r) := \frac{1}{\pi}\arccos(-r) + \frac{1}{\pi}\sqrt{\frac{1+r}{1-r}}\log\left(\frac{2}{1+r}\right)$$

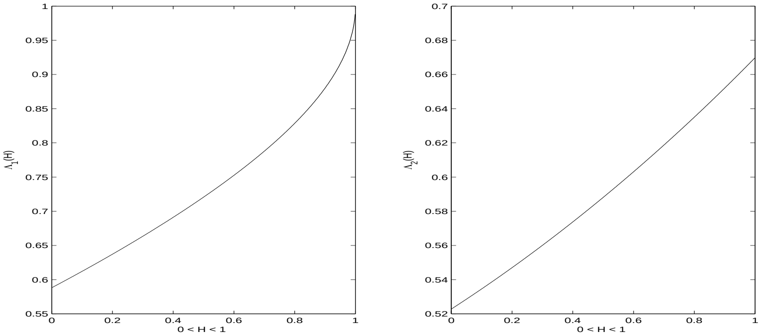
and

$$\rho_L(H) = \begin{cases} 2^{2H-1} - 1 & \text{if } L = 1 \\ \dfrac{-3^{2H} + 2^{2H+2} - 7}{8 - 2^{2H+1}} & \text{if } L = 2 \end{cases}$$

The maps $H \mapsto \rho_L(H)$ and $\rho \mapsto \lambda_0(\rho)$ are non decreasing.

For $L = 1$ or 2, we define

$$\lambda_L(H) = \lambda_0(\rho_L(H)).$$



Complicated formulas for $\gamma^{(IR)}(H)$

- For a fBm with Hurst index H, we have

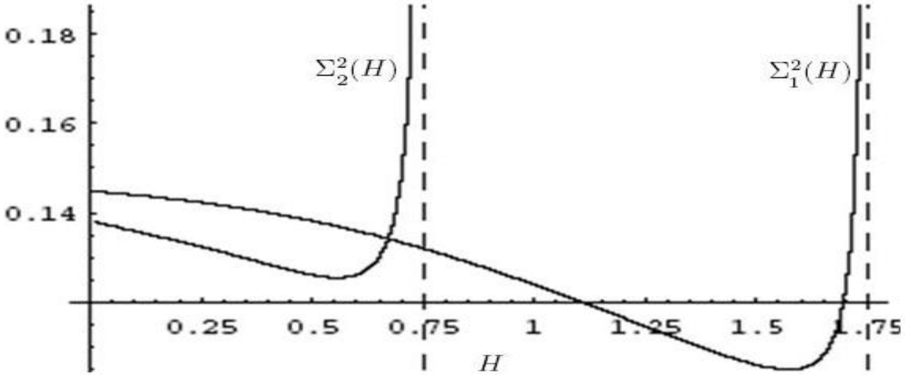
$$(\lambda_L)^{-1}(IRS_{L,n}) \to H \qquad \textit{when } n \to \infty.$$

- Formulas for $\gamma^{(IR)}(H)$

$$\begin{aligned}
\gamma^{(IR_2)}(H) &= \left[\frac{\partial}{\partial x}(\lambda_2)^{-1}(\lambda_2(H))\right]^2 \times \sigma(H)^2 \\
\sigma(H)^2 &= \sum_{k \in \mathbf{Z}} cov\left(\psi_0, \psi_k\right) \\
\psi_k &= \psi(\delta_2(B_H(k), \delta_2(B_H(k+1))
\end{aligned}$$

To sum up, we can compute $\gamma^{(GQV)}(H)$

The maps $H \mapsto \sigma_L(H)$ can be computed.



Surgailis, D., Teyssiere, G., and Vaiciulis, M. The increment ratio statistic. *J. Multivariate Anal* 99 (2008), 510–541.

Consequences of functional CLT

Functional CLT implies

1. $\boldsymbol{E}\widehat{H}_n^{(GQV)}(t) \to H(t)$, since $\mathbb{G}^{(GQV)}(t)$ zero mean process.
2. $\boldsymbol{E}\widehat{H}_n^{(IR)}(t) \to H(t)$, since $\mathbb{G}^{(IR)}(t)$ zero mean process
3. the speed of convergence is $\sqrt{2\varepsilon_n \cdot n}$
4. We can also deduce confidence intervals.
5. Integral Square Error (after rescaling) will converge to $$\int_0^1 \gamma^{(GQV)}(H(t))\,dt \neq 0, \quad \text{resp.} \quad \int_0^1 \gamma^{(IR)}(H(t))\,dt \neq 0.$$

. . .illustrated by simulations

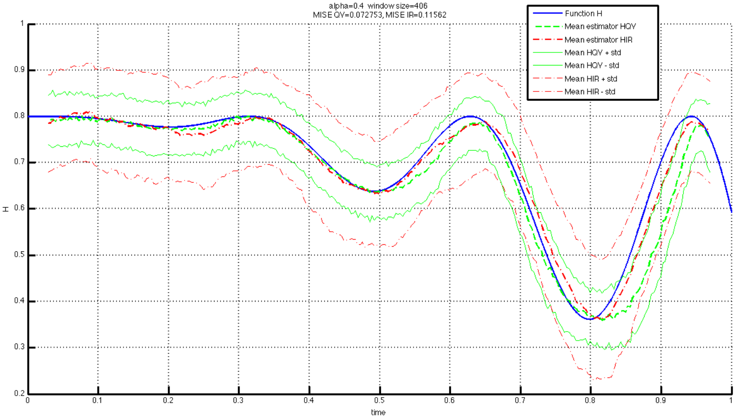


FIGURE – Mean estimation of regular Hurst index $H(t)$, Software by JM Bardet

Paths of $\widehat{H}(t)$, for $H(t)$ regular

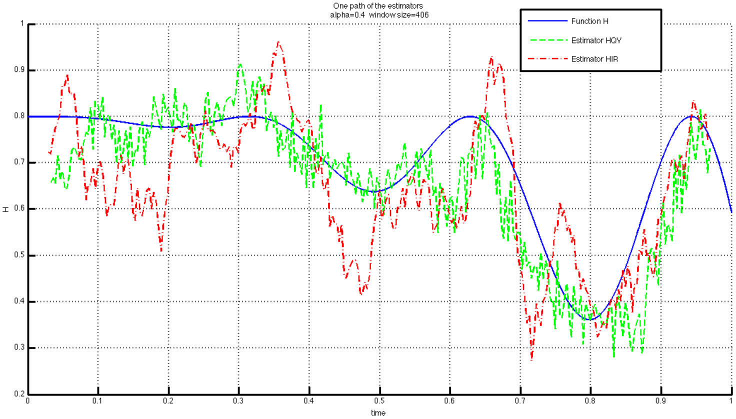
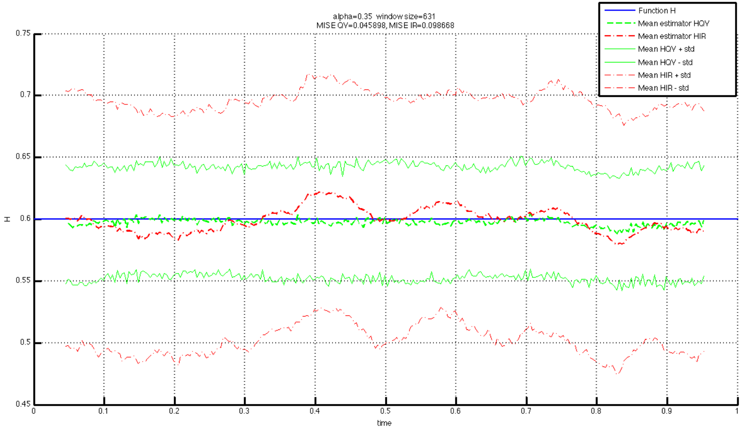


FIGURE – path of the estimation of a regular Hurst index $H(t)$, Software by JM Bardet

Simulations of mean $\widehat{H}(t)$ for $H(t) = 0.6$ (fBm)



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Paths of $\widehat{H}(t)$ for $H(t) = 0.6$ (fBm)

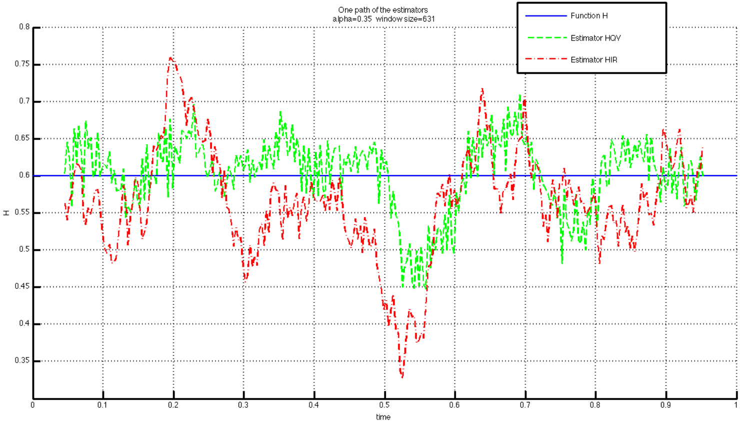


FIGURE – path of the estimation of a regular Hurst index $H(t)$, Software by JM Bardet

Simulations of mean $\widehat{H}(t)$ for $H(t)$ fBm 0.35

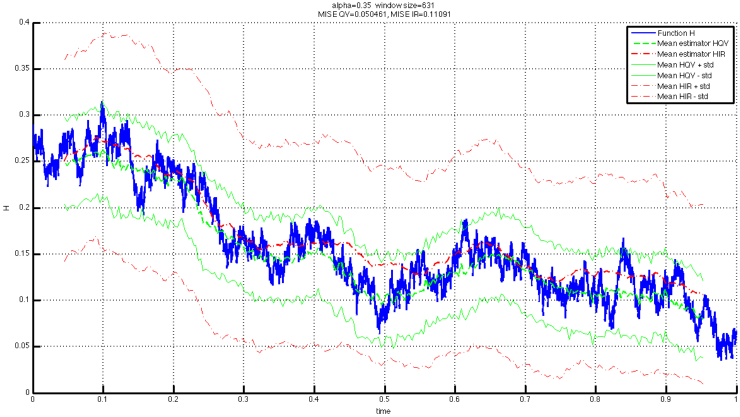


FIGURE – Mean estimation of regular Hurst index $H(t)$, Software by JM Bardet

Paths of $\widehat{H}(t)$ for $H(t)$ fBm 0.35

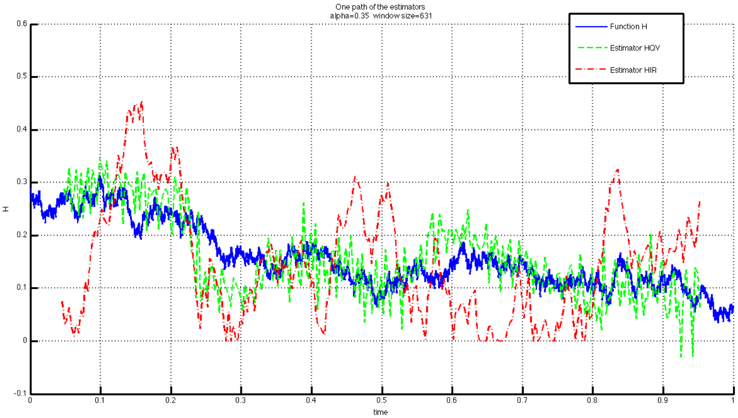


FIGURE – path of the estimation of a regular Hurst index $H(t)$, Software by JM Bardet

Explanation of the statistical artifact

In this covariance structure, we have

$$\mathrm{cov}(\mathbb{G}'(t_1), \mathbb{G}'(t_2)) = 0$$

for all $(t_1, t_2) \in (0,1)^2 \quad \text{such that} \quad t_1 \neq t_2$. This explains the statistical artifact

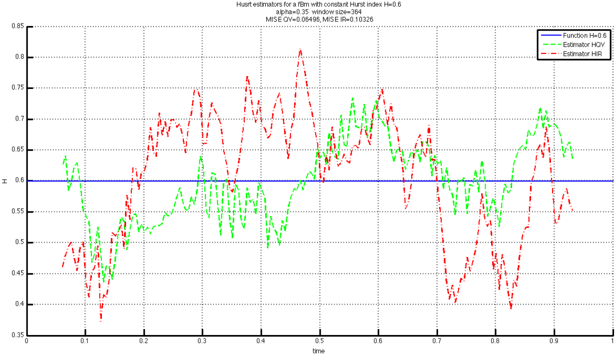
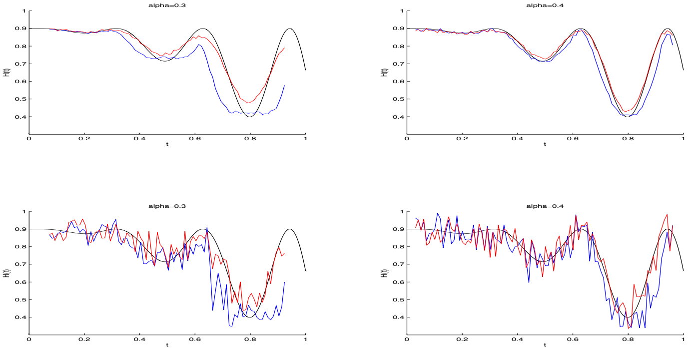
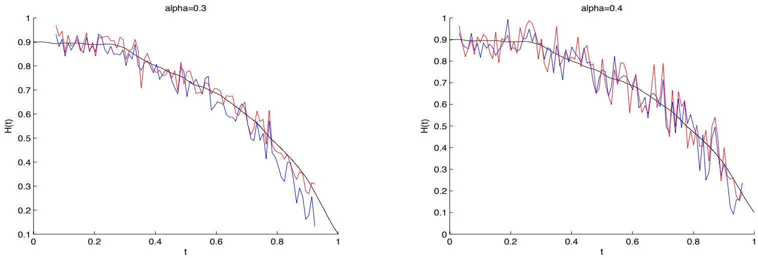


Illustration by Fig.1 p.1022 in Bardet-Surgailis, SPA (2013).



Estimates of the function $H_4(t) = 0.1 + 0.8(1-t)\sin^2(10t)$ with $t \in (0,1)$ for $n = 6000$ and $\alpha = 0.3$ and $\alpha = 0.4$ (from left to right). The top row represents the mean trajectories of $\widehat{H}(t)$ and $H^{(IR2)}(t)$ the localized IRS estimator obtained from 100 independent replications of MBM with the above function $H(\cdot)$.
The bottom row represents a trajectory of $\widehat{H}(t)$ and $H^{(IR2)}(t)$ obtained from one trajectory of MBM with the above function $H(\cdot)$. The graphs of $H(t)$, $\widehat{H}(t)$ and $H^{(IR2)}(t)$ are in black, blue and red, respectively.

Fig.2 p.1023 in Bardet-Surgailis, SPA (2013).



- Trajectories of $\widehat{H}(t)$ and $H^{(IR2)}(t)$ for one of the 50 differentiable Hurst functions $H(\cdot) \in \mathcal{C}^{1.5-}$ for $n = 6000$ and $\alpha = 0.3$ and $\alpha = 0.4$ (from left to right).
- For $\alpha = 0.3$, we have $2\varepsilon_n = 882 \times h_n$.
  For $\alpha = 0.4$, we have $2\varepsilon_n = 370 \times h_n$.
- The graphs of $H(t)$, $\widehat{H}(t)$ and $H^{(IR2)}(t)$ are in black, blue and red, respectively.

Conclusion on estimation of time-varying Hurst index $H(t)$

1. The naive estimators $\widehat{H}^{(GQV)}(t)$ and $\widehat{H}^{(IR)}(t)$ have rough paths, even when $H(t)$ is regular or constant.
2. This statistical artifact is explained by the functional CLT (Coeurjolly, 2005 or Bardet-Surgailis, 2013).

Convergence of the normalized square error

Proposition

From the previous functional CLT, we deduce the convergence of normalized square error

$$\frac{\left\{(2n\varepsilon_n)\times\left[\frac{1}{n}\sum_{k=1}^{n}\left|\widehat{H}_n(t_k)-H(t_k)\right|^2\right]-\int_0^1\gamma\big(H(t)\big)\,dt\right\}}{\left[\left(\frac{2}{n}\right)\times\int_0^1\gamma\big(H(t)\big)^2\,dt\right]^{1/2}}\quad\xrightarrow[N\to\infty]{\mathcal{D}}\quad\mathcal{N}(0,1),$$

1. where $H(t)$ is the right Hurst index,
2. and $\gamma(H)$ is given by the functional CLT for $\widehat{H}^{(GQV)}(t)$, resp. $\widehat{H}^{(IR)}(t)$.

A fitting test for time-varying Hurst index

We want to test if a time-varying Hurst index $\widetilde{H}(\cdot)$ is an admissible model, that is

$$(H_0) : \widetilde{H}(\cdot) = H(\cdot) \qquad \textit{versus} \qquad (H_1) : \widetilde{H}(\cdot) \neq H(\cdot).$$

We use the test statistic

$$T_n(\widetilde{H}) = \frac{(2n\varepsilon_n) \times \left[\frac{1}{n}\sum_{k=1}^{n} |\widehat{H}_n(t_k) - \widetilde{H}(t_k)|^2\right] - \int_0^1 \gamma\big(\widetilde{H}(t)\big)\, dt}{\left(\left(\frac{2}{n}\right) \times \int_0^1 \gamma\big(\widetilde{H}(t)\big)^2\, dt\right)^{1/2}}.$$

A fitting test for time-varying Hurst index

- Under the null hypothesis, we have

$$T_n(\widetilde{H}) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

- On the other hand, we cannot calculate the power of the test since $H(\cdot) \in \mathcal{C}([0,1])$ which is an infinite dimensional vector space.

1st application to model rejection

The naive time-varying estimator of the Hurst index could not be chosen as a valid model. Let

$$\widetilde{H}(t) = \lim_{n\to\infty} \widehat{H}_n(t)$$

Then

$$\begin{aligned} T_n(\widetilde{H}(t)) &\simeq \frac{-\int_0^1 \gamma_{\widetilde{H}(t)} dt}{\left(\frac{2}{n}\int_0^1 (\gamma_{\widetilde{H}(t)})^2 dt\right)^{1/2}} \\ &\simeq -\sqrt{\frac{n}{2}} \times \frac{\|\gamma_{\widetilde{H}(t)}\|_{L^1(]0;1[)}}{\|\gamma_{\widetilde{H}(t)}\|_{L^2(]0;1[)}} \longrightarrow \infty \text{ as } n \to \infty \end{aligned}$$

The null hypothesis (H_0) is asymptotically rejected.

A real dataset (West Texas Intermediate, Oil price)

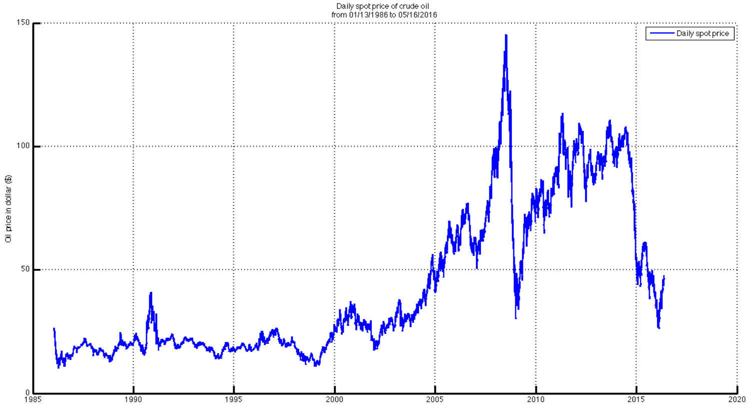


FIGURE – Daily spot price of WTI oil from January 1986 to May 2016

Estimation of the Hurst index

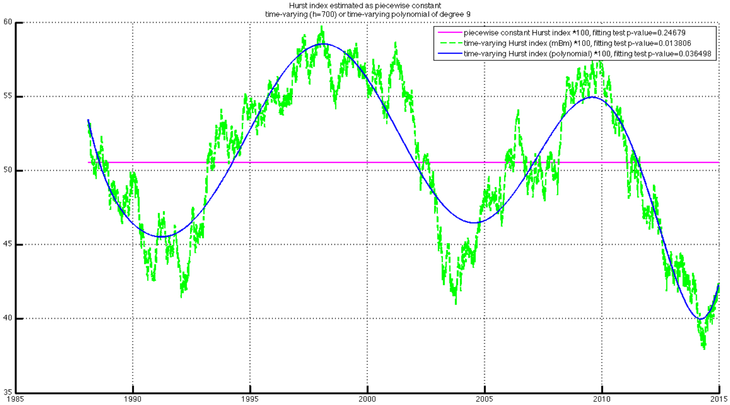


FIGURE – Three estimators of the Hurst index

Result of the test

1. We reject the second and third estimator
2. We accept the model of constant estimator with $H(t) = 0.5052$.

Toward a model selection for Hurst index

We compute T_{obs} and the corresponding p-value for the following models

- X fBm, i.e. $H(t)$ constant (green cross)
- X step-fBm, i.e. $H(t)$ piecewise constant (magenta cross) ;
- X mBm with $H(t)$ a polynomial function (blue line) :
  For $degree = 1 \ldots 15$,
  we compute T_{obs} and the corresponding p-value.
- X mBm with $H(t)$ the naive estimator $\widehat{H}^{(IR)}(t)$ (red cross)

Selection of the Hurst index

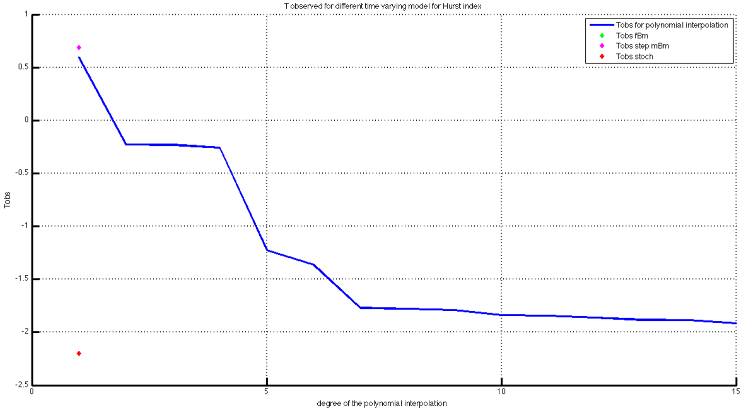


FIGURE – T_{obs} for the different models of Hurst index

Selection of the Hurst index

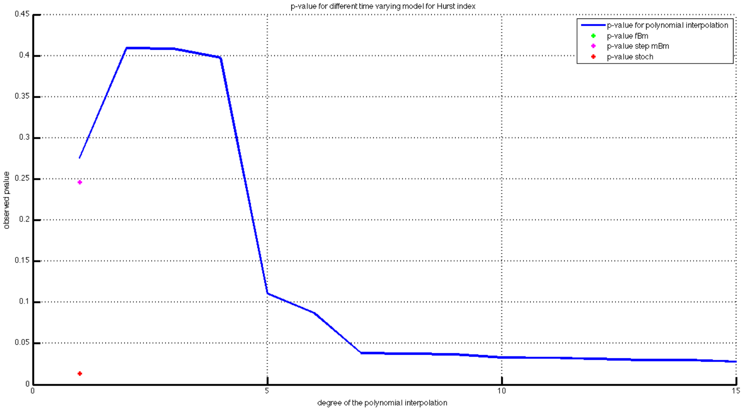


FIGURE – *p*-value for the different models of Hurst index

A real dataset : NASDAQ (1971-2009)

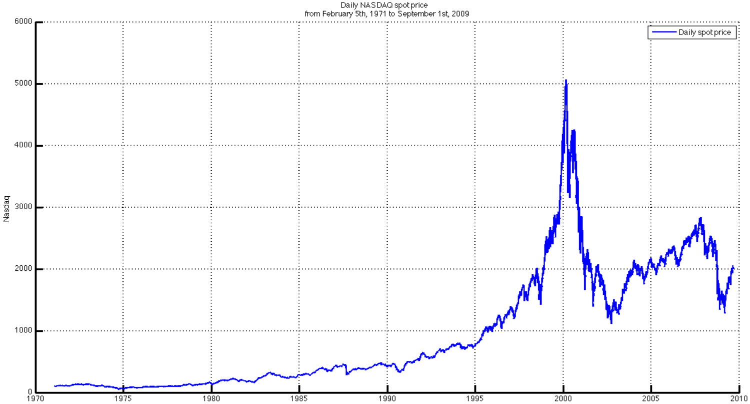


FIGURE – Daily NASDAQ spot price from 1971 to 2009

log NASDAQ (1971-2009)

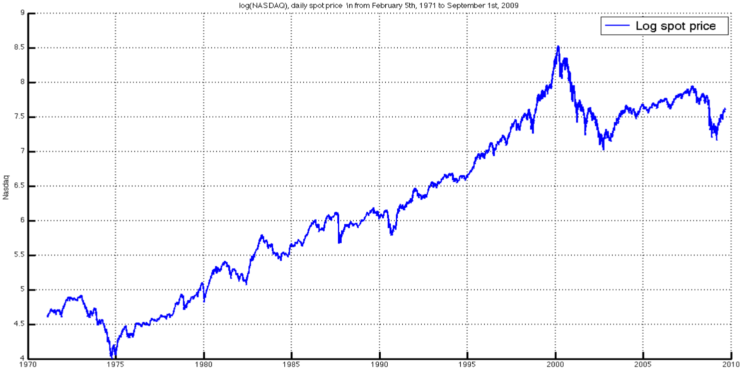


FIGURE – Log NASDAQ daily price from 1971 to 2009. From 1975 to 1999 the Nasdaq increased yearly by 10.5%.

Selection of the Hurst index for NASDAQ, T_{obs}

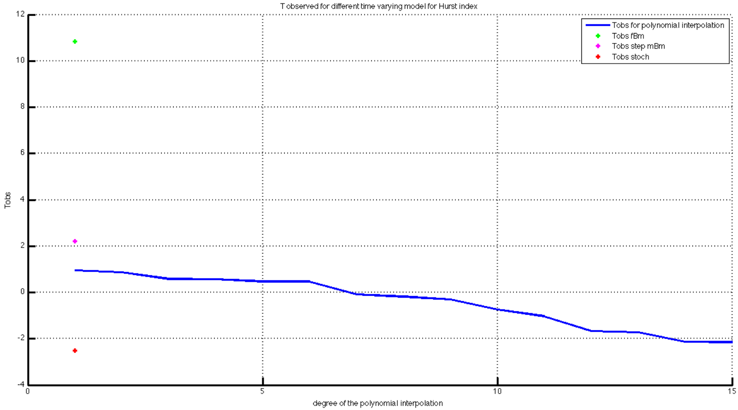


FIGURE – T_{obs} for the different models of Hurst index

Selection of the Hurst index for NASDAQ, *p*-values

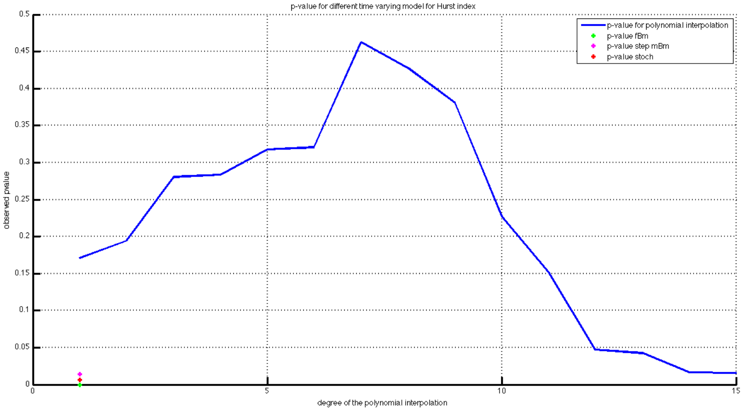
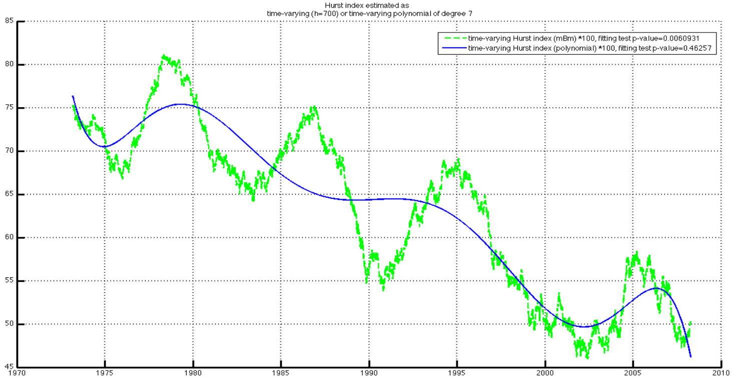


FIGURE – *p*-value for the different models of Hurst index

And the winner is...
$H(t)$ a polynomial function with degree 7
(Hurst index for NASDAQ)



Chocolate medal :
$H(t)$ a polynomial function with degree 5

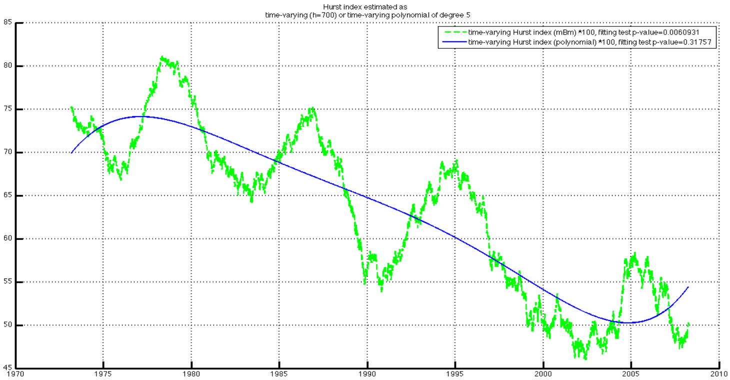


FIGURE – Hurst index for NASDAQ, as a polynomial function of degree 5

At the very beginning Nasdaq has a Hurst index around $H = 0.7$, but it decreases to $H = 1/2$ when the market becomes more and more liquid.

A real dataset : Heartrate series for a Marathon runner

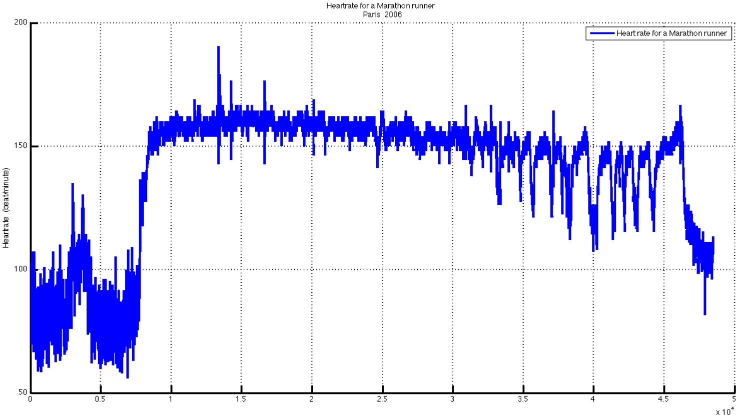


FIGURE – Heart rate for a Marathon runner (Paris 2006)

Selection of the Hurst index for Heart rate (Marathon runner), T_{obs}

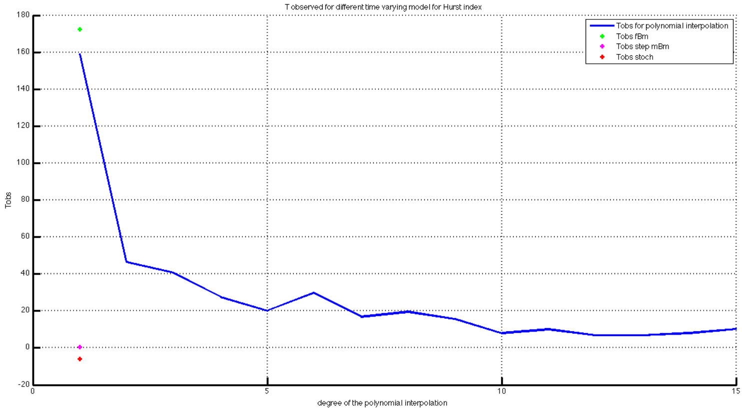


FIGURE – T_{obs} for the different models of Hurst index

Selection of the Hurst index for Heart rate (Marathon runner), *p*-values

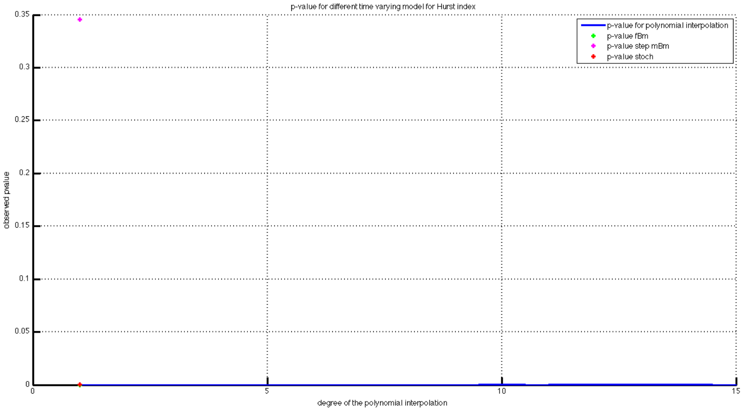


FIGURE – *p*-value for the different models of Hurst index

The winner is alone : $H(t)$ piecewise constant (i.e. X Step-fBm)

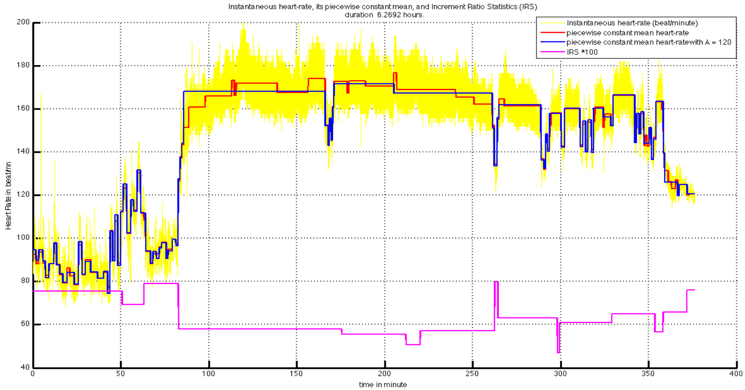


FIGURE – Piecewise constant mean and Hurst index of Marathon runner heart rate.

Another Marathon runner

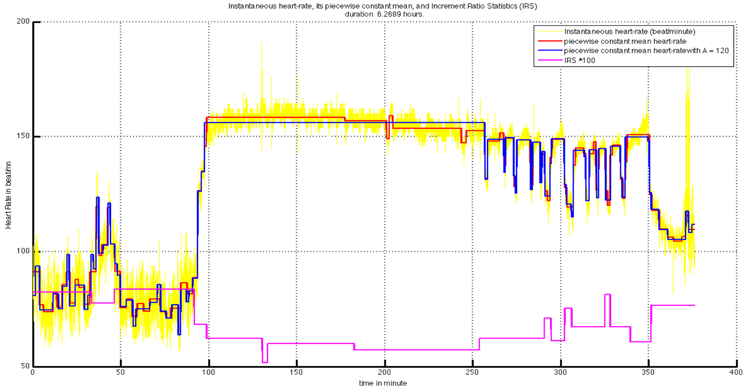


FIGURE – Yellow=Observed Heart rate. Blue : Change detection on the mean with $\delta f = 7$ beat/mn and automatic choice of window size. Red : Change detection on the mean with $\delta f = 10$ beat/mn and window size=2 minutes. Magenta : $\lambda(\rho_2(H))$

A shiftworker, playing soccer in the afternoon

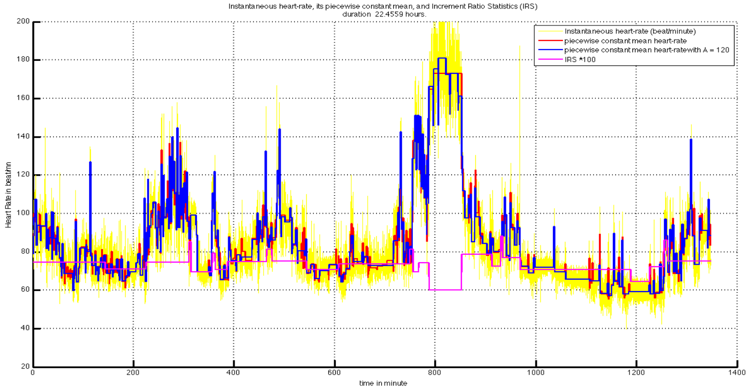
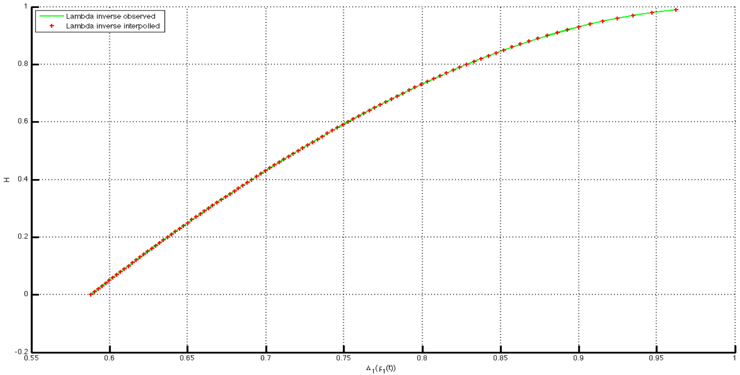


FIGURE – Yellow=Observed Heart rate. Blue : Change detection on the mean with $\delta f = 7$ beat/mn and automatic choice of window size. Red : Change detection on the mean with $\delta f = 10$ beat/mn and window size=2 minutes. Magenta : $\lambda(\rho_1(H))$

From $\lambda(\rho_1(H))$ to H



Polynomial interpolation of $\lambda(\rho_1(H)) \longmapsto H$

A shiftworker, playing soccer in the afternoon

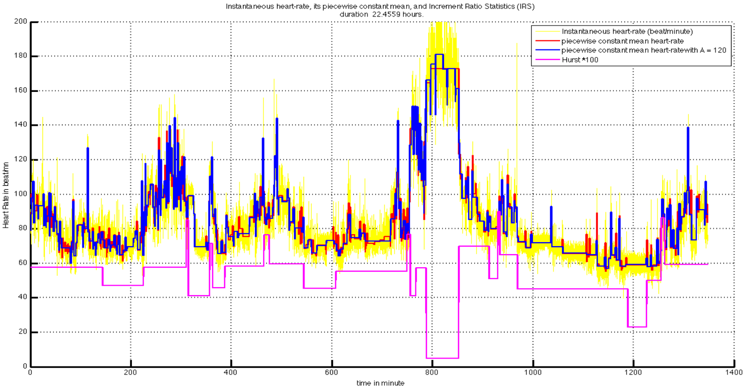


FIGURE – Yellow=Observed Heart rate. Blue : Change detection on the mean with $\delta f = 7$ beat/mn and automatic choice of window size. Red : Change detection on the mean with $\delta f = 10$ beat/mn and window size=2 minutes. Magenta : Hurst index

Conclusion

1. We have explained the statistical artifact.
2. We propose a fitting test for admissible time-varying Hurst index $H(t)$.
3. Selection of the best model should be enhanced.

Remerciements

- This work has been supported by grant ANR-12-BS01-0016-01 entitled *"Do Well B."* and a regional grant PEPS (2015-16).

- see `http://math.univ-bpclermont.fr/DoWellB/index-fr.html`
- the video `http://videocampus.univ-bpclermont.fr/?v=miuSSU1YSeye`
- and the article `http://m.huffpost.com/fr/entry/10115348`

Thank you for your interest ☺

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