On Parameter Estimation and Hyphothesis Testing for Poisson Processes in Case of a Change-point with Variable Jump Size

(joint work with Lin Yang)

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Model of observations

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Recall: $X = \left(X(t),\ t \geqslant 0\right)$ is an (inhomogeneous) Poisson process with intensity function $\lambda(t)$, $t \geqslant 0$, if $X_0 = 0$ and the increments of X on disjoint intervals are independent Poisson random variables:

$$\mathbf{P}\{X(t) - X(s) = k\} = \frac{\left(\int_s^t \lambda(t)\,\mathrm{d}t\right)^k}{k!} \exp\left\{-\int_s^t \lambda(t)\,\mathrm{d}t\right\}.$$

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Asymptotics: $n \to \infty$.

Remark: equivalent to periodic observation on $[0, n\tau]$ and to one observation on $[0, \tau]$ with intensity function $n\lambda_\vartheta^{(n)}(t)$, $0 \leqslant t \leqslant \tau$.

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(C1) The intensity function $\lambda_\vartheta^{(n)}(t)$ can be written as $\lambda_\vartheta^{(n)}(t) = \psi_n(t) + r_n \mathbb{1}_{\{t>\vartheta\}}$, where ψ_n is continuous on $[0, \tau]$.

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(C4) There exist some constants $\ell, L > 0$ such that $\ell \leqslant \lambda_\vartheta^{(n)}(t) \leqslant L$ for all $n \in \mathbb{N}$, $\vartheta \in (\alpha,\beta)$ and $t \in [0,\tau]$.

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(C1) – **(C3)** and $r > -\min\limits_{t\in[0,\tau]} \psi(t) \quad \Longrightarrow \quad$ **(C4)** for n sufficiently large.

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Remark:

(C1) – (C2) $\quad\Longleftarrow\quad$ **(C0)** The intensity function $\lambda_\vartheta^{(n)}(t)$ can be written as $\lambda_\vartheta^{(n)}(t) = \psi(t) + r_n \mathbb{1}_{\{t>\vartheta\}}$, where $\psi > 0$ is continuous on $[0,\tau]$.

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The *likelihood* is a (random) càdlàg function of $\vartheta \in (\alpha, \beta)$:

$$L_n\big(\vartheta, X^{(n)}\big) = \exp\left\{ \sum_{j=1}^{n} \int_{[0,\tau]} \ln \lambda_\vartheta^{(n)}(t)\, \mathrm{d}X_j^{(n)}(t) - n \int_0^\tau \big[\lambda_\vartheta^{(n)}(t) - 1\big] \mathrm{d}t \right\}.$$

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The *normalized likelihood ratio* is a process with trajectories in $\mathscr{D}_0(\mathbb{R})$:

$$Z_{n,\vartheta}(v) = \frac{L_n\bigl(\vartheta_v, X^{(n)}\bigr)}{L_n\bigl(\vartheta, X^{(n)}\bigr)}, \qquad v \in \Theta_n = \bigl(\varphi_n^{-1}(\alpha - \vartheta), \varphi_n^{-1}(\beta - \vartheta)\bigr),$$

where $\vartheta_v \triangleq \vartheta + v\varphi_n$, and *normalization rate* $\varphi_n = \varphi_n(\vartheta) \searrow 0$ suitably.

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The process Z_0 is a process with trajectories in $\mathscr{C}_0(\mathbb{R}) \subset \mathscr{D}_0(\mathbb{R})$ defined by

$$Z_0(v) = \exp\left\{W(v) - \frac{|v|}{2}\right\},$$

where W is a standard two-sided Brownian motion (Wiener process).

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Theorem. — Let the conditions **(C1)** – **(C4)** be fulfilled. Then, uniformly in ϑ on any compact set $\mathbb{K} \subset \Theta$, the process $Z_{n,\vartheta}$ converges weakly in the space $\mathscr{D}_0(\mathbb{R})$ to

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Remark: in the case $r \neq 0$, the limit is the same as in the case of a fixed jump size ($r_n \equiv r$) studied by Kutoyants (1984, 1998) $\big($see also Dachian, Kutoyants, Yang (2015) for hypotheses testing$\big)$.

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$$\max\Big\{L_n\big(\widehat{\vartheta}_n+, X^{(n)}\big), L_n\big(\widehat{\vartheta}_n-, X^{(n)}\big)\Big\} = \sup_{\vartheta\in(\alpha,\beta)} L_n\big(\vartheta, X^{(n)}\big).$$

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The *Bayes estimator (BE)* $\widetilde{\vartheta}_n$ (for square loss and) for a given *prior density* p is defined by

$$\widetilde{\vartheta}_n = \frac{\int_\alpha^\beta \vartheta\, p(\vartheta) L_n\big(\vartheta, X^{(n)}\big)\,\mathrm{d}\vartheta}{\int_\alpha^\beta p(\vartheta) L_n\big(\vartheta, X^{(n)}\big)\,\mathrm{d}\vartheta}\,.$$

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Definition. — We say that an estimator ϑ_n^* is asymptotically efficient if

$$\lim_{\delta\to 0}\ \lim_{n\to+\infty}\ \sup_{|\vartheta-\vartheta_0|<\delta} \frac{n^2 r_n^4}{\psi^2(\vartheta)}\,\mathbf{E}_\vartheta^{(n)}(\vartheta_n^*-\vartheta)^2 = \mathbf{E}\zeta^2 = 16\,\boldsymbol{\zeta}(3),$$

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In particular, the relative asymptotic efficiency of $\widehat{\vartheta}_n$ is

$$\frac{\mathbf{E}\zeta^2}{\mathbf{E}\xi^2} = \frac{16\, \boldsymbol{\zeta}(3)}{26} \approx 0.74.$$

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In particular, $\widetilde{\vartheta}_n$ is asymptotically efficient.

Hypothesis testing

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$$\begin{aligned} \mathscr{H}_0 \quad &: \quad \vartheta = \vartheta_0, \\ \mathscr{H}_1 \quad &: \quad \vartheta > \vartheta_0. \end{aligned}$$

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$$\begin{aligned} \mathscr{H}_0 \quad &: \quad \vartheta = \vartheta_0, \\ \mathscr{H}_1 \quad &: \quad \vartheta > \vartheta_0. \end{aligned}$$

Tests: a (randomized) test $\bar\psi_n = \bar\psi_n\bigl(X^{(n)}\bigr)$ is defined as the probability to reject the hypothesis $\mathscr{H}_0$.

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Pitman's approach

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Close or contiguous alternatives: $\vartheta = \vartheta_u \triangleq \vartheta_0 + u\varphi_n$, where $\varphi_n = \varphi_n(\vartheta_0) \searrow 0$ and $u \in \Theta_n^+ = \left[0, \varphi_n^{-1}(b - \vartheta_0)\right)$.

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$$Z_{n,\vartheta_0}(v) = \frac{L_n\left(\vartheta_0 + v\varphi_n, X^{(n)}\right)}{L_n\left(\vartheta_0, X^{(n)}\right)}, \qquad v \in \Theta_n^+,$$

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Limits of the likelihood ratio

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Limit under hypothesis: under $\vartheta = \vartheta_0$, the limit of the normalized likelihood ratio Z_{n,ϑ_0} is

$$Z_0(v) = \exp\left\{W(v) - \frac{v}{2}\right\}, \qquad v \geqslant 0,$$

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Limit under alternative: under $\vartheta = \vartheta_u$ (with any fixed $u > 0$), the limit of the normalized likelihood ratio Z_{n,ϑ_0} is

$$Z_0^{(u)}(v) = \exp\left\{W(v) - \frac{|v-u|}{2} + \frac{u}{2}\right\}, \qquad v \geqslant 0,$$

where W is a standard Brownian motion (Wiener process).

Wald's test

Wald's test

The maximum likelihood estimator (MLE) $\widehat{\vartheta}_n$ is now given by

$$\max\left\{L_n\left(\widehat{\vartheta}_n+, X^{(n)}\right), L_n\left(\widehat{\vartheta}_n-, X^{(n)}\right)\right\} = \sup_{\vartheta\in[\vartheta_0,b)} L_n\left(\vartheta, X^{(n)}\right).$$

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The Wald's test (WT) is based on the MLE $\widehat{\vartheta}_n$ and is defined by

$$\phi_n^{\circ}\big(X^{(n)}\big) = \mathbb{1}_{\left\{\frac{n\,r_n^2}{\psi(\vartheta_0)}(\widehat{\vartheta}_n-\vartheta_0)>m_\varepsilon\right\}}$$

with m_ε solution of

$$\int_{m_\varepsilon}^{+\infty}\left(\frac{1}{\sqrt{2\pi t}}\exp\{-t/8\} - \frac{1}{2}\,\Phi\big(-\sqrt{t}/2\big)\right)\mathrm{d}t = \varepsilon,$$

where Φ is the distribution function of $\mathcal{N}(0,1)$.

Properties of the WT

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- The test ϕ_n° belongs to $\mathcal{K}_\varepsilon$.

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- The power function of ϕ_n° converges to $\beta^\circ(u)$:

$$\beta(\phi_n^\circ, u) \longrightarrow \beta^\circ(u),$$

where

$$\beta^\circ(u) = \mathbf{P}\left\{\xi_u > m_\varepsilon - u\right\}, \qquad Z_0(\xi_u) = \sup_{v \geqslant -u} Z_0(v)$$

and/or

$$\beta^\circ(u) = \mathbf{P}\Big\{\xi_+^{(u)} > m_\varepsilon\Big\}, \qquad Z_0^{(u)}\Big(\xi_+^{(u)}\Big) = \sup_{v \geqslant 0} Z_0^{(u)}(v).$$

General likelihood ratio test

General likelihood ratio test

The general likelihood ratio test (GLRT) is defined by

$$\widehat{\phi}_n\left(X^{(n)}\right) = \mathbb{1}_{\left\{Q(X^{(n)})>1/\varepsilon\right\}}$$

with

$$\begin{aligned} Q\left(X^{(n)}\right) &= \sup_{\vartheta\in[\vartheta_0,b)} \frac{L_n\left(\vartheta, X^{(n)}\right)}{L_n\left(\vartheta_0, X^{(n)}\right)} \\ &= \frac{\max\left\{L_n\left(\widehat{\vartheta}_n+, X^{(n)}\right), L_n\left(\widehat{\vartheta}_n-, X^{(n)}\right)\right\}}{L_n\left(\vartheta_0, X^{(n)}\right)}\,. \end{aligned}$$

Properties of the GLRT

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- The test $\widehat{\phi}_n$ belongs to $\mathcal{K}_\varepsilon$.

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where

$$\widehat{\beta}(u) = \mathbf{P}\{\widehat{Z}_u > 1/\varepsilon\}, \qquad \widehat{Z}_u = \big(Z_0\,(-u)\big)^{-1} \sup_{v \geqslant -u} Z_0(v)$$

and/or

$$\widehat{\beta}(u) = \mathbf{P}\{\widehat{Z}_+^{(u)} > 1/\varepsilon\}, \qquad \widehat{Z}_+^{(u)} = \sup_{v \geqslant 0} Z_0^{(u)}(v).$$

First Bayes test

First Bayes test

The *Bayes estimator (BE)* $\widetilde{\vartheta}_n$ (for square loss and) for a given *prior density* p is now given by

$$\widetilde{\vartheta}_n = \frac{\int_{\vartheta_0}^{b} \vartheta\, p(\vartheta) L_n\big(\vartheta, X^{(n)}\big)\,\mathrm{d}\vartheta}{\int_{\vartheta_0}^{b} p(\vartheta) L_n\big(\vartheta, X^{(n)}\big)\,\mathrm{d}\vartheta}\,.$$

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The first Bayes test (BT1) is a Wald-type test based on the BE $\widetilde{\vartheta}_n$ and is defined by

$$\widetilde{\phi}_n\big(X^{(n)}\big) = \mathbb{1}_{\left\{\frac{n\, r_n^2}{\psi(\vartheta_0)}(\widetilde{\vartheta}_n - \vartheta_0) > k_\varepsilon\right\}}$$

with k_ε solution of

$$\mathbf{P}\left\{\zeta_+ > k_\varepsilon\right\} = \varepsilon, \qquad \zeta_+ = \frac{\int_0^{+\infty} v\, Z_0(v)\,\mathrm{d}v}{\int_0^{+\infty} Z_0(v)\,\mathrm{d}v}\,.$$

Properties of the BT1

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- The test $\widetilde{\phi}_n$ belongs to $\mathcal{K}_\varepsilon$.

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- The test $\widetilde{\phi}_n$ belongs to $\mathcal{K}_\varepsilon$.
- The power function of $\widetilde{\phi}_n$ converges to $\widetilde{\beta}(u)$:

$$\beta(\widetilde{\phi}_n, u) \longrightarrow \widetilde{\beta}(u),$$

where

$$\widetilde{\beta}(u) = \mathbf{P}\left\{\zeta_u > k_\varepsilon - u\right\}, \qquad \zeta_u = \frac{\int_{-u}^{+\infty} v\, Z_0(v)\,\mathrm{d}v}{\int_{-u}^{+\infty} Z_0(v)\,\mathrm{d}v}$$

and/or

$$\widetilde{\beta}(u) = \mathbf{P}\{\zeta_+^{(u)} > k_\varepsilon\}, \qquad \zeta_+^{(u)} = \frac{\int_0^{+\infty} v\, Z_0^{(u)}(v)\,\mathrm{d}v}{\int_0^{+\infty} Z_0^{(u)}(v)\,\mathrm{d}v}.$$

Second Bayes test

Second Bayes test

For a test $\bar{\phi}_n$, the mean (or averaged) power is

$$\beta(\bar{\phi}_n) = \int_{\vartheta_0}^{b} \beta(\bar{\phi}_n, \vartheta)\, p(\vartheta)\mathrm{d}\vartheta.$$

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The second Bayes test (BT2) is the test which maximizes the mean power and is defined by

$$\widetilde{\phi}_n^\star\big(X^{(n)}\big) = \mathbb{1}_{\left\{R(X^{(n)}) > g_\varepsilon\right\}}$$

with

$$R\big(X^{(n)}\big) = \frac{n\, r_n^2 \int_{\vartheta_0}^{b} L_n\big(\vartheta, X^{(n)}\big)\, p(\vartheta)\, \mathrm{d}\vartheta}{p(\vartheta_0)\, \psi(\vartheta_0)\, L_n\big(\vartheta_0, X^{(n)}\big)}$$

and g_ε solution of

$$\mathbf{P}\{\widetilde{Z}_+ > g_\varepsilon\} = \varepsilon, \qquad \widetilde{Z}_+ = \int_0^{+\infty} Z_0(v)\, \mathrm{d}v.$$

Properties of the BT2

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- The test $\widetilde{\phi}_n^\star$ belongs to $\mathcal{K}_\varepsilon$.

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- The test $\widetilde{\phi}_n^\star$ belongs to $\mathcal{K}_\varepsilon$.
- The power function of $\widetilde{\phi}_n^\star$ converges to $\widetilde{\beta}^\star(u)$:

$$\beta(\widetilde{\phi}_n^\star, u) \longrightarrow \widetilde{\beta}^\star(u),$$

where

$$\widetilde{\beta}^\star(u) = \mathbf{P}\{\widetilde{Z}_u > g_\varepsilon\}, \qquad \widetilde{Z}_u = \big(Z_0(-u)\big)^{-1} \int_{-u}^{+\infty} Z_0(v)\,\mathrm{d}v$$

and/or

$$\widetilde{\beta}^\star(u) = \mathbf{P}\{\widetilde{Z}_+^{(u)} > g_\varepsilon\}, \qquad \widetilde{Z}_+^{(u)} = \int_0^{+\infty} Z_0^{(u)}(v)\,\mathrm{d}v.$$

Numerical simulations

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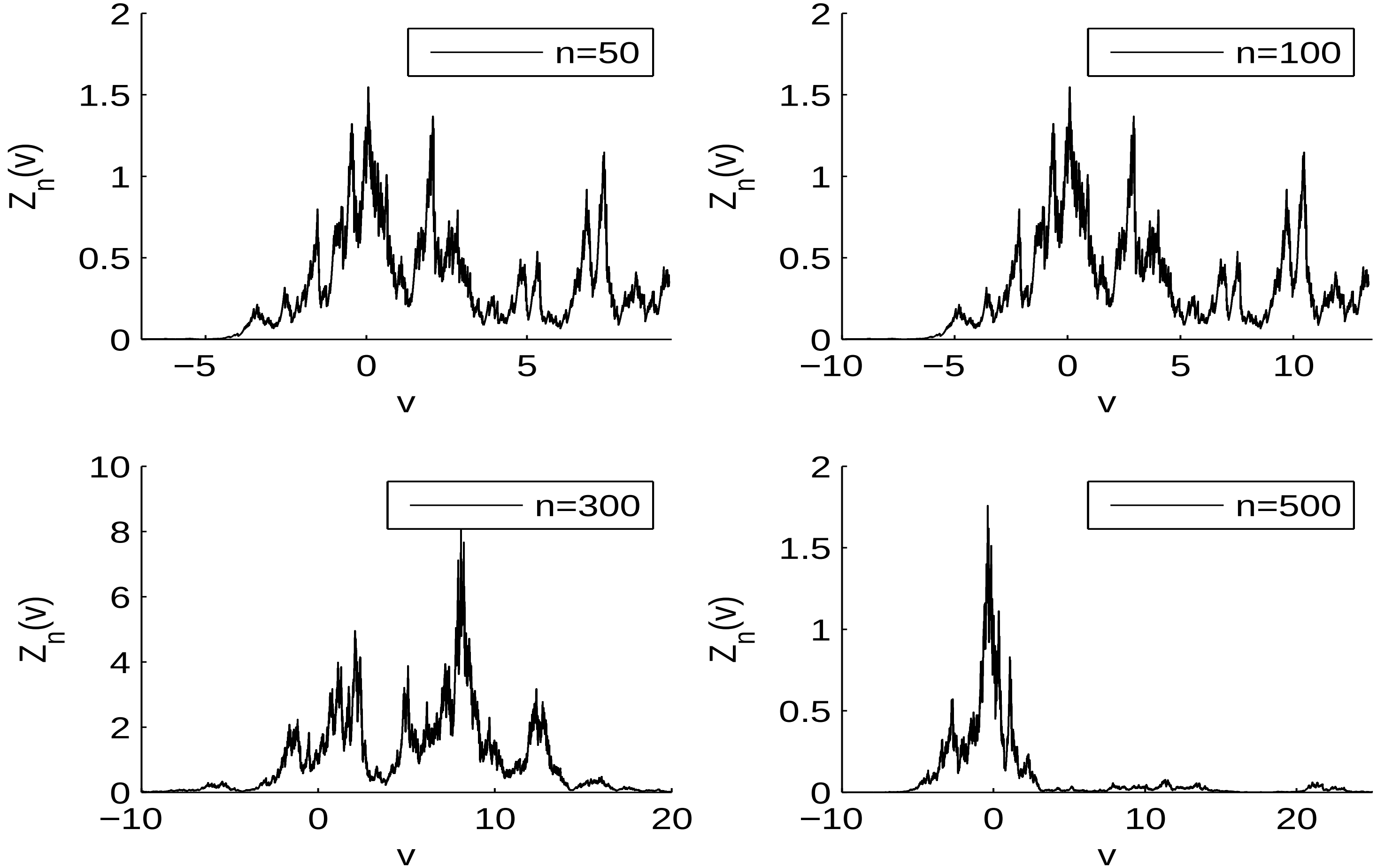
- prior distribution for BT1: uniform on $[2, 4)$.
- some thresholds for WT and BT1:

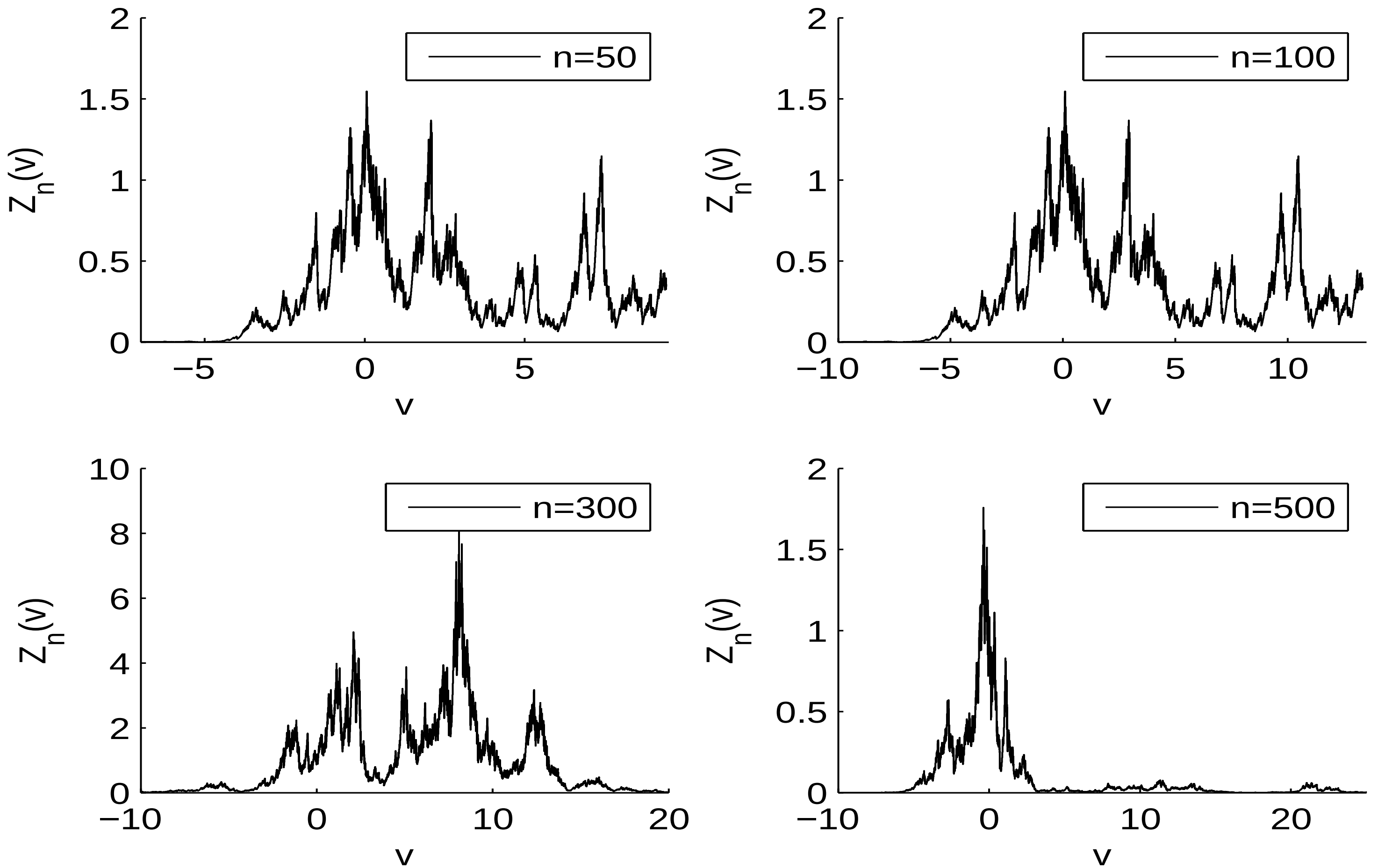
ε	0.001	0.005	0.01	0.05	0.1	0.2
m_ε	30.336	20.686	14.886	7.282	4.531	2.236
k_ε	24.877	17.588	16.782	8.582	5.573	3.024

Likelihood ratio

Likelihood ratio



Likelihood ratio



Some realization of Z_{n,ϑ_0}

Convergence of power functions

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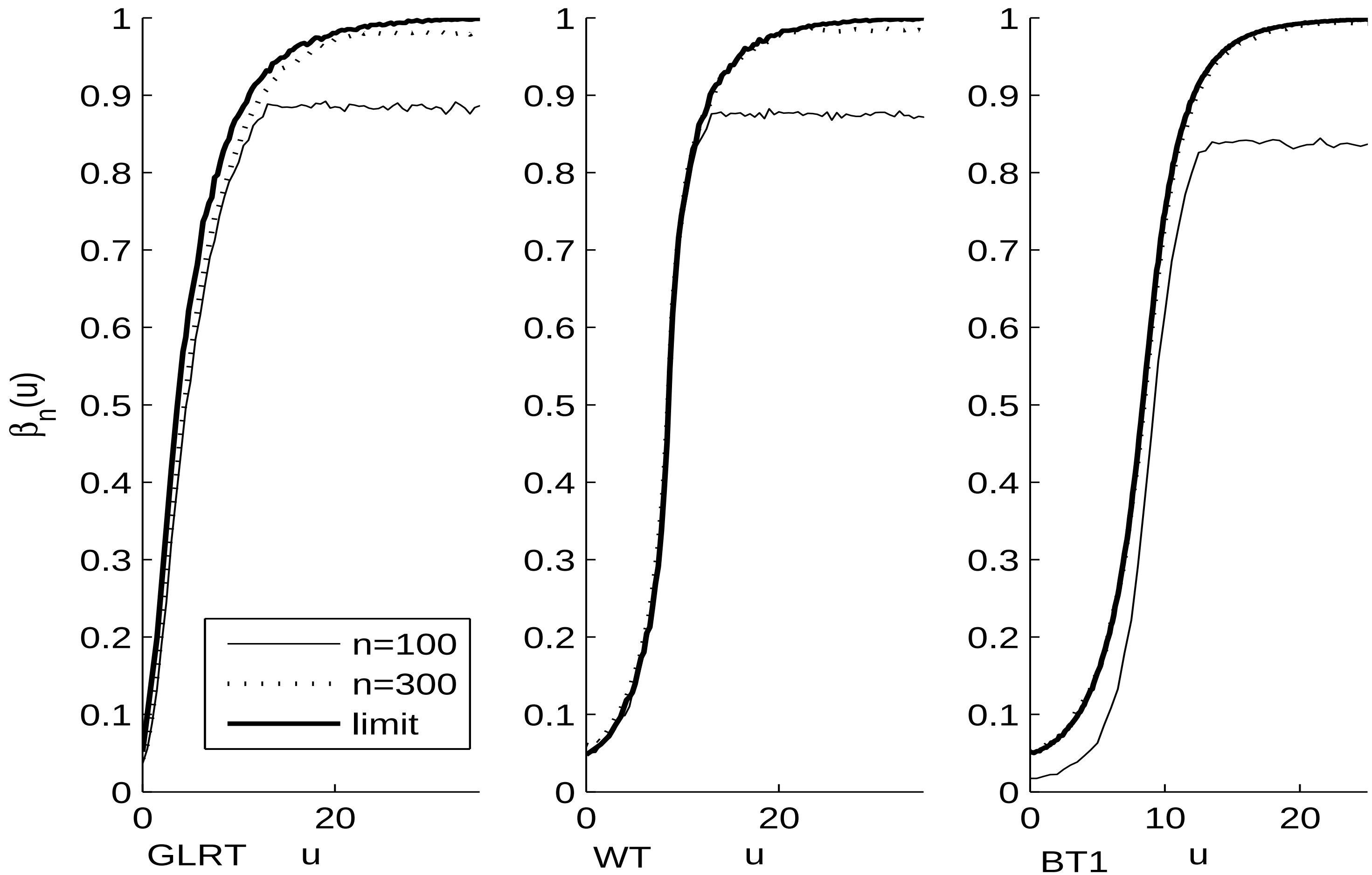

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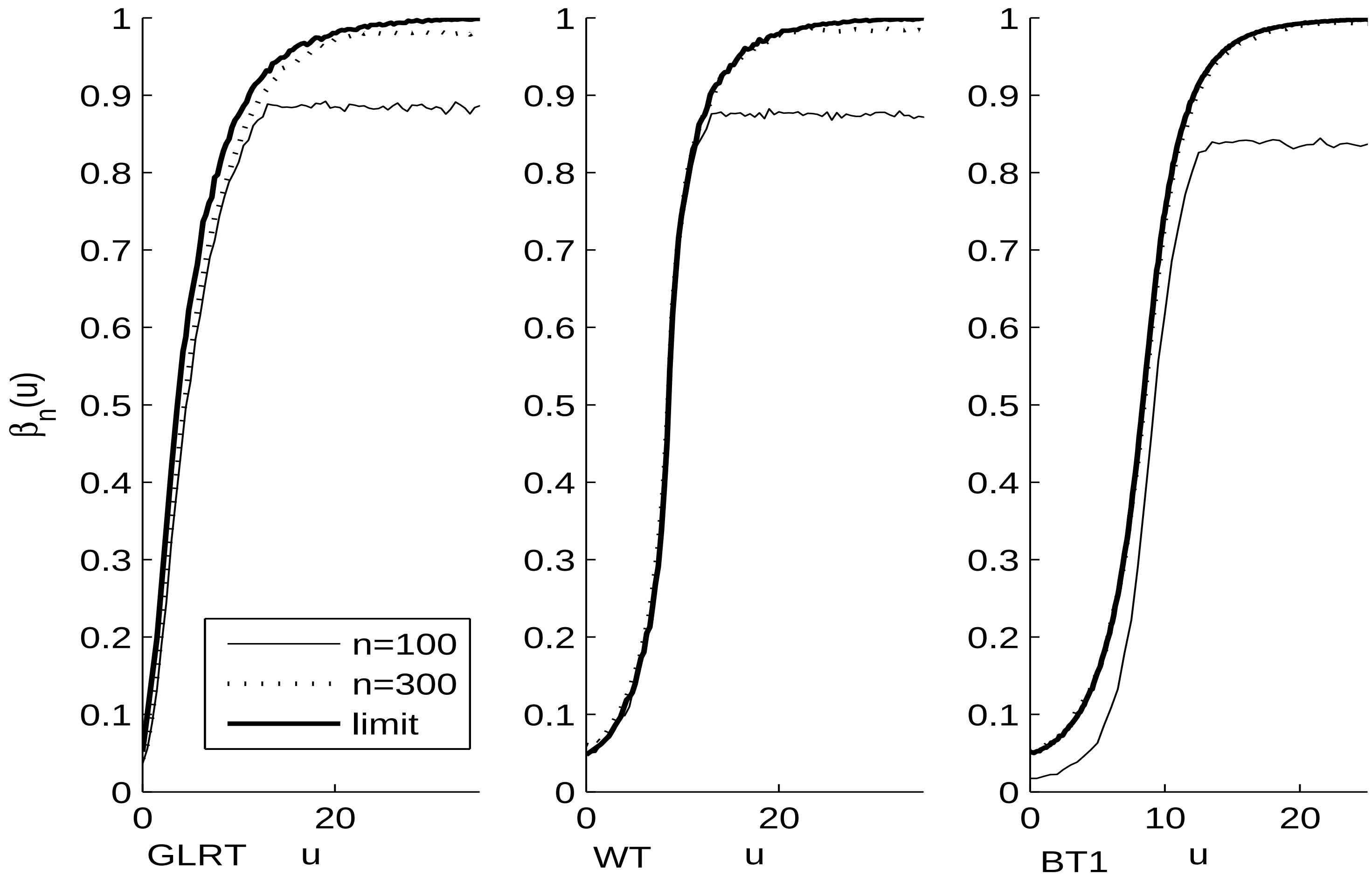

Convergence of power functions of the GLRT, of the WT and of the BT1

Comparison of limit power functions

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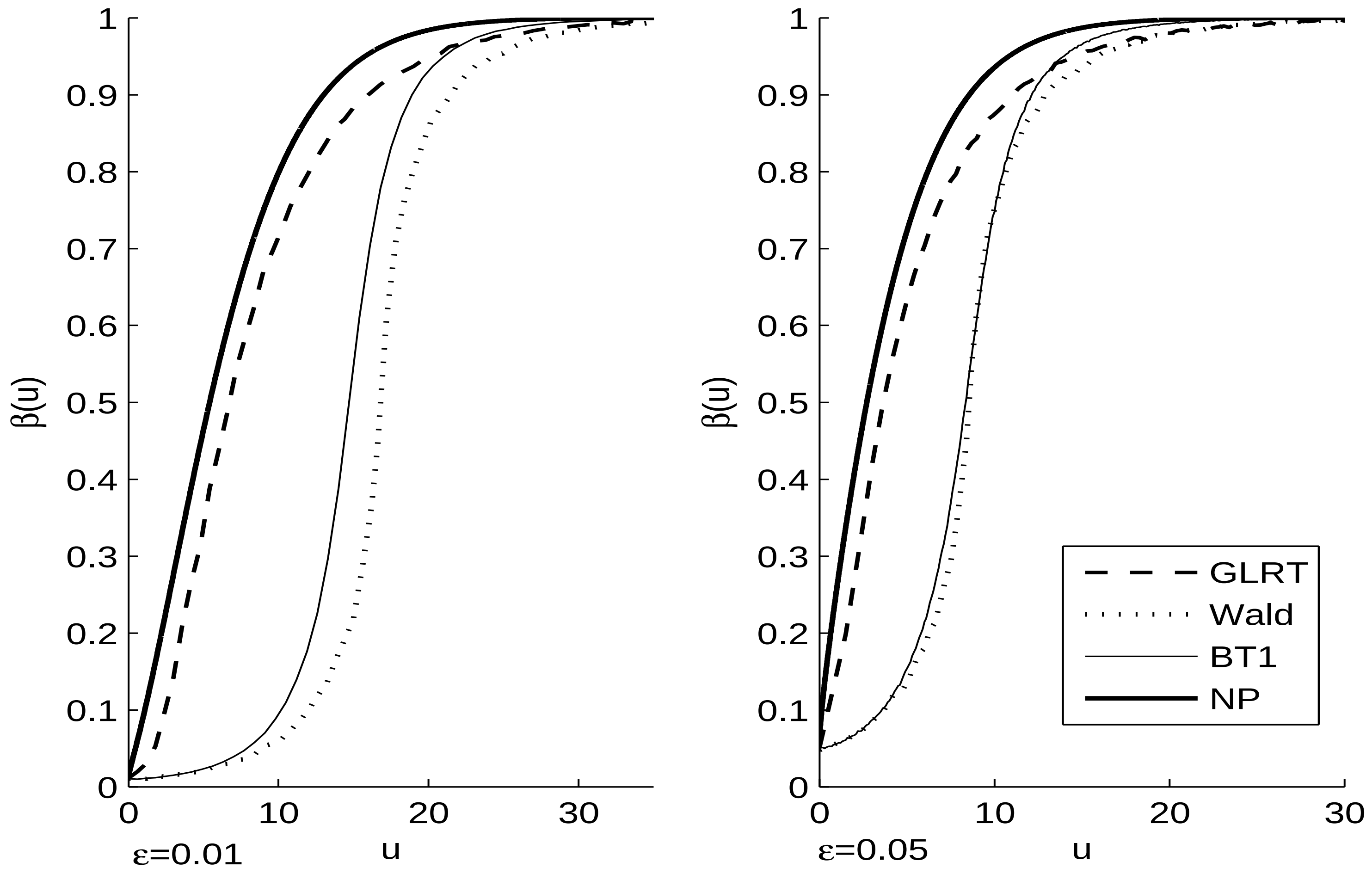

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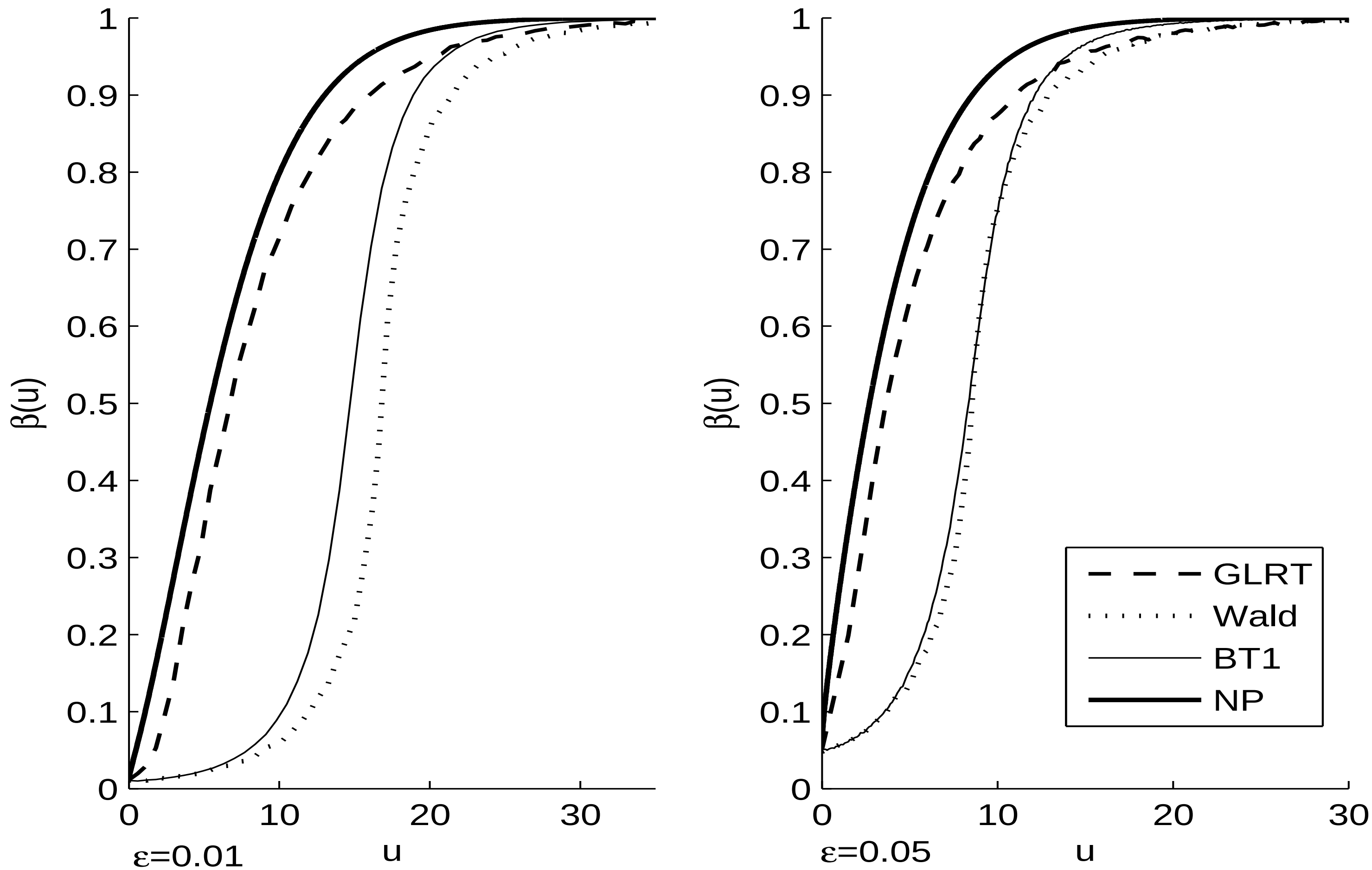

Comparison of limit power functions with the Neyman-Pearson envelope

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