Local limit theorem in non-Gaussian quasi-likelihood inference

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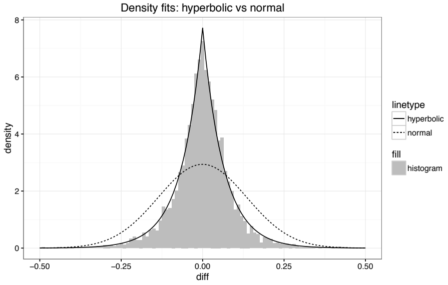
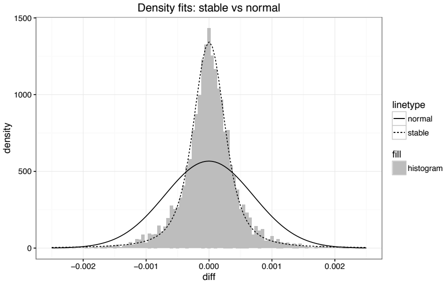
1 Background and objective

2 Sufficient conditions for stable local limit theorem

3 Summary

- NYSE: IBM stock data (©CREST JST)





Primary objective: L^1-local limit theorem

- **Standard β-stable Lévy process** $J = (J_t)$: $\mathcal{L}(J_1) = S_\beta$, i.e.

$$\mathbb{E}(e^{iuJ_t}) = e^{-t|u|^\beta} = \exp\left(t \int (\cos(uz) - 1)\, \frac{c_\beta}{|z|^{\beta+1}} dz\right)$$

  - Scaling property: $h^{-1/\beta} J_h \sim S_\beta$, smooth density ϕ_β

- **Locally (small-time) standard β-stable Lévy process** $J = (J_t)$:

$$\mathcal{L}(h^{-1/\beta} J_h) \Rightarrow S_\beta, \qquad h \to 0$$

  - $\mathcal{L}(h^{-1/\beta} J_h)$ admits a b'dd. conti. density f_h (Bertoin and Doney, 1997)
  - Roughly, e.g. the Lévy density: $z \mapsto \frac{c_\beta}{|z|^{\beta+1}} \{1 + o(1)\}$ for $|z| \to 0$

Easy conditions for "$\lim_{h \to 0} \frac{1}{\sqrt{h}} \int |f_h(y) - \phi_\beta(y)| dy = 0$" ?

Why?

Inference for SDE from high-frequency data

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dJ_t$$

- Estimate true $\theta_0 = (\alpha_0, \gamma_0)$ from $(X_{t_j})_{j=0}^n$, where $t_j = jh_n, \quad h_n := T/n$
- J is a locally β-stable pure-jump $(1 \le \beta < 2)$ Lévy process
  - Small-time non-Gaussian, infinite-activity and/or infinite-variation character

- Technical merits
  - Consistent estimation of stochastic location-scale structure over fixed time domain
  - Sidestep most stability constraints: stationarity, ergodicity, finite variance, ...

Stable quasi-likelihood: heuristic

- Euler (small-time) approximation:

$$
\begin{aligned}
\Delta_j X := X_{t_j} - X_{t_{j-1}} &\overset{\mathbb{P}_\theta}{=} \int_{t_{j-1}}^{t_j} a(X_s,\alpha)ds + \int_{t_{j-1}}^{t_j} c(X_{s-},\gamma)dJ_s \\
&\overset{\mathbb{P}_\theta}{\approx} a(X_{t_{j-1}},\alpha)h_n + c(X_{t_{j-1}},\gamma)(J_{t_j} - J_{t_{j-1}}) \\
&=: a_{j-1}(\alpha)h_n + c_{j-1}(\gamma)\Delta_j J
\end{aligned}
$$

- $\epsilon_{nj}(\theta) := \dfrac{\Delta_j X - a_{j-1}(\alpha)h_n}{h_n^{1/\beta}c_{j-1}(\gamma)} \sim$ i.i.d. S_β, approximately

Stable Quasi-Maximum Likelihood Estimator $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\gamma}_n)$; SQMLE

$$
\hat{\theta}_n = (\hat{\alpha}_n, \hat{\gamma}_n) \in \underset{\theta\in\Theta}{\operatorname{argmax}} \sum_{j=1}^n \log\left\{ \frac{1}{h_n^{1/\beta}c_{j-1}(\gamma)} \phi_\beta\big(\epsilon_{nj}(\theta)\big) \right\}
$$

Asymptotic mixed normality of SQMLE

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dJ_t, \quad (X_{jT/n})_{j=0}^n$$

Theorem. Under some assumptions ($\beta \ge 1$; next slide),

$$\begin{pmatrix} n^{1/\beta - 1/2}(\hat{\alpha}_n - \alpha_0) \\ \sqrt{n}(\hat{\gamma}_n - \gamma_0) \end{pmatrix} \xrightarrow{\mathcal{L}} MN\left(0, \mathrm{diag}[\Sigma_{T,\alpha}(\theta_0)^{-1}, \Sigma_{T,\gamma}(\theta_0)^{-1}]\right)$$

$$\Sigma_{T,\alpha}(\theta_0) := T^{2(1-1/\beta)} \frac{1}{T}\int_0^T \frac{\{\partial_\alpha a(X_t, \alpha_0)\}^{\otimes 2}}{c(X_t, \gamma_0)^2} dt \cdot \int \frac{\{\partial \phi_\beta(y)\}^2}{\phi_\beta(y)} dy,$$

$$\Sigma_{T,\gamma}(\theta_0) := \frac{1}{T}\int_0^T \frac{\{\partial_\gamma c(X_t, \gamma_0)\}^{\otimes 2}}{c(X_t, \gamma_0)^2} dt \cdot \int \frac{\{\phi_\beta(y) + y\partial \phi_\beta(y)\}^2}{\phi_\beta(y)} dy$$

- Masuda, H. (2017), Non-Gaussian quasi-likelihood estimation of SDE driven by locally stable Lévy process. arXiv:1608.06758 (v3)
- Asymptotically efficient in several cases (maybe in general).
  - Locally stable Lévy process: Ivanenko, Kulik and M (2015)
  - SDE: Clément and Gloter (2015); Clément, Gloter and Nguyen (2017)

Assumptions

$$dX_t = a(X_t,\alpha)dt + c(X_{t-},\gamma)dJ_t, \quad (X_{jT/n})_{j=0}^n$$

1. Regularity of the coefficients
   - (a,c) smooth enough, with $a(\cdot,\alpha_0)$ and $c(\cdot,\gamma_0)$ globally Lipschitz.
   - $\exists K \ge 0,\ \forall x,\ \sup_\gamma |c(x,\gamma)|^{-1} \le K(1+|x|)^K$

2. Identifiability
   - $\mathbb{P}_{\theta_0}\left\{ \begin{pmatrix} a(X_t,\alpha) \\ c(X_t,\gamma) \end{pmatrix}_{t\le T} = \begin{pmatrix} a(X_t,\alpha_0) \\ c(X_t,\gamma_0) \end{pmatrix}_{t\le T} \right\} = 1 \quad \Rightarrow \quad \theta = \theta_0$

3. **Driving-noise structure** for the pdf f_h of $\mathcal{L}(h^{-1/\beta}J_h)$
   - $1 \le \beta < 2$
   - $\exists \epsilon > 0,\ \int |y|^{\beta-\epsilon}|f_h(y) - \phi_\beta(y)|dy \to 0$
   - $\sqrt{n}\int |f_h(y) - \phi_\beta(y)|dy \to 0$ for the pdf f_h of $\mathcal{L}(h^{-1/\beta}J_h)$

- How to verify in an easy manner?:
  - $\exists \epsilon > 0, \ \int |y|^{\beta-\epsilon} |f_h(y) - \phi_\beta(y)| dy \to 0$
  - $\sqrt{n} \int |f_h(y) - \phi_\beta(y)| dy \to 0$
- Here we forget that $1 \le \beta < 2$, required to handle Euler-approx. error.

Sufficient conditions I

(L) J's Lévy measure $\nu(dz) = g(z)dz$ for a symmetric g s.t.

- $g(z) = \dfrac{c_\beta}{|z|^{\beta+1}} \{1 + \rho(z)\}$ $(z \neq 0)$;
- $\exists \delta, \epsilon_\rho > 0,\ \exists c_\rho \geq 0,\ \forall |z| \in (0, \epsilon_\rho], \quad |\rho(z)| \leq c_\rho |z|^\delta$

$$\text{(L)} + \sup_z |\rho(z)| < \infty \Rightarrow \exists \epsilon > 0, \ \int |y|^{\beta-\epsilon} |f_h(y) - \phi_\beta(y)| dy \to 0$$

(G1) $\rho \in \mathcal{C}^1(\mathbb{R} \setminus \{0\})$ and the pair (c_ρ, β, δ) satisfies either

- $c_\rho = 0$, or
- $c_\rho > 0$ and $\delta > \beta$ with $|\rho(z)| + |z \partial \rho(z)| \leq c_\rho |z|^\delta$ $(z \neq 0)$

$$\text{(L)} + \text{(G1)} + \mathrm{supp}(g) \subset [-K, K] \ (^\exists K > 0) \Rightarrow \sqrt{n} \int |f_h(y) - \phi_\beta(y)| dy \to 0$$

- Additional conditions on ρ lose no generality through the localization.

Sufficient conditions II

(L) J's Lévy measure $\nu(dz) = g(z)dz$ for a symmetric g s.t.

- $g(z) = \dfrac{c_\beta}{|z|^{\beta+1}}\{1+\rho(z)\}$ $(z \neq 0)$;
- $\exists \delta, \epsilon_\rho > 0,\ \exists c_\rho \geq 0,\ \forall |z| \in (0, \epsilon_\rho],\quad |\rho(z)| \leq c_\rho |z|^\delta$

(G2) $\psi_h \in \mathcal{C}^1(\mathbb{R} \setminus \{0\})$, and for $\psi_h(u) := h \log \mathbb{E}(e^{iuh^{-1/\beta}J_h})$ and $\varphi_0(u) := e^{-|u|^\beta}$:

- $\exists c_\psi \geq 0,\quad |\partial_u \psi_h(u)| \lesssim \dfrac{1}{u} \vee u^{c_\psi} \quad (u > 0)$;
- $\exists \epsilon_\psi(h) \to 0,\ \exists r \in [0,1]$ s.t.
  - ★ $\displaystyle\int_{(0,\infty)} u^r \varphi_0(u) \left|\partial_u \psi_h(u) + \beta u^{\beta-1}\right| du \leq \epsilon_\psi(h)$
  - ★ $\sqrt{n}(\epsilon_\psi(h) \vee h^{a_\nu})^{\frac{\beta}{\beta+r}} \to 0$ for $a_\nu := \begin{cases} 1 & (c_\rho = 0) \\ (\delta/\beta) \wedge 1 & (c_\rho > 0). \end{cases}$

$$(\mathrm{L}) + (\mathrm{G2}) + \sup_z |\rho(z)| < \infty \Rightarrow \sqrt{n} \int |f_h(y) - \phi_\beta(y)| dy \to 0$$

Two examples satisfying "(L) & (G2)", but not "(L) & (G1)"

- **Exponentially tempered stable** with index $\beta \in [1, 2)$:

$$\psi_h(u) = \begin{cases} \dfrac{1}{\pi}\left\{\lambda h \log\left(1 + \dfrac{u^2}{\lambda^2 h^2}\right) - 2u \arctan\left(\dfrac{u}{\lambda h}\right)\right\} & (\beta = 1) \\ 2c_\beta \Gamma(-\beta)\left[(\lambda^2 h^{2/\beta} + u^2)^{\beta/2} \cos\left\{\beta \arctan\left(\dfrac{u}{\lambda h^{1/\beta}}\right)\right\} - \lambda^\beta h\right] & (\beta \in (1, 2)) \end{cases}$$

  - $\displaystyle\int_{(0,\infty)} \varphi_0(u)|\partial_u \psi_h(u) + \beta u^{\beta-1}|du \lesssim h^{1\wedge(1/\beta)} = h^{1/\beta}$

- **Generalized hyperbolic** $(\beta = 1)$ except for the variance gamma:

$$\partial_u \psi_h(u) + 1 = 1 - \frac{u}{\sqrt{(\eta h)^2 + u^2}} \frac{K_{\lambda+1}}{K_\lambda}\left(\frac{1}{h}\sqrt{(\eta h)^2 + u^2}\right)$$

  - $\displaystyle\int_{(0,\infty)} \varphi_0(u)\,|\partial_u \psi_h(u) + 1|\,du \lesssim \begin{cases} h \log(1/h) & (\lambda \neq -1/2) \\ h & (\lambda = -1/2: \text{ NIG case}) \end{cases}$

Summary

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dJ_t, \quad (X_{jT/n})_{j=0}^n$$
$$\mathcal{L}(h^{-1/\beta} J_h) \Rightarrow S_\beta$$

- Sufficient conditions for the local limit theorems $(1 \le \beta < 2)$:
  - $\exists \epsilon > 0, \ \int |y|^{\beta-\epsilon} |f_h(y) - \phi_\beta(y)| dy \to 0$
  - $\sqrt{n} \int |f_h(y) - \phi_\beta(y)| dy \to 0$ for the pdf f_h of $\mathcal{L}(h^{-1/\beta} J_h)$

- In terms of either
  - Lévy density $g(z)$ of J, or
  - Lévy-Khintchine exponent $u \mapsto h \log \mathbb{E}(e^{iuh^{-1/\beta} J_h})$

- For verifying the key assumptions in quasi-likelihood inference for (α, γ).
  - Local limit results + Coefficients' regularity $\Rightarrow$ Asymptotic mixed normality
  - No moment conditions and no ergodicity

Some references

- **Masuda, H. (2017), Non-Gaussian quasi-likelihood estimation of SDE driven by locally stable Lévy process. arXiv:1608.06758 (v3)**

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